Graph Theory – CMSC-27500 – Spring 2015
Instructor: Laszlo Babai
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #12.   Posted 5-12, 8:15pm
Due Thursday, May 14, typeset in **LaTeX**.

Read the homework instructions on the website. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a **separate sheet**, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. As before, $G = (V, E)$ denotes a graph or digraph with n vertices and m edges.

We say that two paths connecting vertices u and v are *internally disjoint* if they share no vertices other than u and v. The "vertex-disjoint" versions of Menger's Theorem talk about pairwise internally disjoint paths.

Let $k \geq 1$. We say that the graph G is *k-connected* if for every pair of vertices $u \neq v$ there exist k internally disjoint paths between u and v. The *connectivity* $\kappa(G)$ is the largest k such that G is k-connected. ("κ" is the Greek letter "kappa." In Latex: `\kappa`.)

The d-dimensional cube Q_d is the Cartesian product of d copies of K_2. (So Q_d has 2^d vertices and $d2^{d-1}$ edges.)

12.1 DO: Review the proof of the following theorem. If a every edge in a network has unit capacity and the value of the maximum flow is k then there exist k edge-disjoint $s \to t$ paths.

12.2 DO: Review the proofs of the various versions of Menger's Theorem discussed in class.

12.3 DO: Prove the undirected vertex-disjoint version of Menger's Theorem by reducing it to the directed vertex-disjoint version.

12.4 DO: Prove: if the graph G is k-connected then $n \geq k + 1$.

12.5 DO (connectivity): Prove: (a) For every graph G, the connectivity $\kappa(G)$ is at most the minimum degree of G. (b) $\kappa(K_n) = n-1$. (c) $\kappa(C_n) = 2$. (d) The connectivity of the $r \times s$ toroidal grid $C_r \Box C_s$ is 4. (Here $r, s \geq 3$.) (e) $\kappa(Q_d) = d$. Here Q_d denotes the d-dimensional cube, see the Preamble.

12.6 DO: Find a connected regular graph G of degree 100 such that $\kappa(G) = 1$.

12.7 DO (cube): Find the following parameters of the d-dimensional cube Q_d: the degree of every vertex, χ, α, ν, τ, the diameter, the average distance (the expected distance between two random vertices), the number of 4-cycles.

12.8 HW (8 points): Let G be a k-connected graph and S a set of k vertices of G. Construct the graph H by adding a new vertex u to G and joining it by an edge to each vertex in S. So H has $n+1$ vertices and $m+k$ edges. Prove: H is k-connected. Use one of Menger's theorems; state the version you use.

12.9 DO: Let G be a k-connected graph and let S and T be disjoint k-subsets of V (i. e., $|S| = |T| = k$). Prove: there exist k disjoint paths, each connecting a vertex of S with a vertex of T. (These paths must be pairwise disjoint (share no vertices), not just internally disjoint, so they will establish a bijection between S and T.)