Graph Theory – CMSC-27500 – Spring 2015
Instructor: Laszlo Babai
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #13. Posted 5-15, 3:45pm
Due Tuesday, May 19, typeset in **LaTeX**.

Instructor will hold a **problem session** Monday, May 18, 4:30–5:20pm, in Ry-276 (optional, will help you prepare for the next quiz). Come prepared with questions.

Read the homework instructions on the website. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a **separate sheet**, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. A *multigraph* $G = (V, E, \psi)$ consists of a set V of vertices, a set E of edges, and an assignment $\psi : E \to \{$unordered pairs of not necessarily distinct vertices$\}$. So $\psi(e) = \{u, v\}$ means that the edge e joins the vertices u and v. If $u = v$ then e is a "loop." If for edges $e, f \in E$ we have $\psi(e) = \psi(f)$ then we say that e and f are parallel edges. We continue to use n to denote the number of vertices and m the number of edges.

Multi-digraphs are defined analogously; here the function $\psi : E \to V \times V$ assigns ordered pairs of vertices to each edge.

We say that the (multi)graph G is *k-connected* between the vertices u and v if there exist k internally disjoint paths between u and v. The *connectivity* of G between u and v, denoted $\kappa(G; u, v)$, is the largest k such that G is k-connected between u and v. Note that $\kappa(G) = \min_{u\neq v} \kappa(G; u, v)$.

"Connectivity" is also referred to as "vertex-connectivity."

We say that the (multi)graph G is *ℓ-edge-connected* between the vertices u and v if there exist k edge-disjoint paths between u and v. The *edge-connectivity* of G between u and v, denoted $\lambda(G; u, v)$, is the largest ℓ such that G is ℓ-edge-connected between u and v. The *edge-connectivity* of G is $\lambda(G) = \min_{u\neq v} \lambda(G; u, v)$.

("λ" is the Greek letter "lambda.")

The directed multigraph versions of these notions of connectivity are defined analogously.

Let H be a multigraph. A *subdivision* of H is obtained by subdividing some of the edges of H by new vertices (which will then have degre 2). A subdivision of H is also called a *topological* H. Notice that the subdivisions of K_2 are the paths and the subdivisions C_1 (a loop) are the cycles. The subdivisions of K_5 and $K_{3,3}$ are called *Kuratowski graphs.* Two multigraphs are said to be *homeomorphic* if they are subdivisions of the same multigraph.

A *contraction* of a graph G is a graph defined in the following way: Let $V = V_1 \cup \cdots \cup V_k$ be a partition of V such that the subgraph induced by each V_i is connected. Let H have vertex set $[k]$; we make i, j adjacent ($1 \le i < j \le k$) if there is an edge between V_i and V_j. So for instance K_5 is a contraction of of the Petersen graph (what is the partition?). Contractions can be obtained by repeatedly contracting an edge. A *minor* of G is a graph isomorphic to a subgraph of a contraction of G.

13.1 DO: Study multigraphs and multi-digraphs from the Bondy–Murty text.

13.2 DO: (a) Extend the concept of a network and network flows to multi-digraphs. (b) Prove the Max-flow min-cut Theorem in this more general context. Do not repeat the proof of the original (non-multi) version; simply reduce the multi version to the non-multi version.

13.3 DO: Extend all the four versions of Menger's Theorem to multi-(di)graphs. Prove them by reduction to the non-multi situation.

13.4 DO (Maximal vs. maximum set of internally disjoint paths): Find a graph with two special vertices, s and t, such that $\kappa(G; s, t) = 100$ but there exists an s–t path P that alone is a maximal set of internally disjoint s–t paths (i. e., no s–t path is internally disjoint from P).

13.5 DO (Min degree vs. min cover in bipartite graphs): (a) Let G be a bipartite graph with bipartition (A, B) (so A and B are the set of red and blue vertices, respectively, in a 2-coloring of G). Let $n_1 = |A|$ and $n_2 = |B|$. Let $k \ge 1$, and assume every vertex of G has degree $\ge k$. Assume further that $n_1 > 2k$ and $n_2 > 2k$. Prove: $\tau(G) \ge 2k$. (b) Prove that the lower bound on τ given in part (a) is tight in the following sense. For every n_0, find a bipartite graph G as above such that (b1) $n_1 \ge n_0$ and $n_2 \ge n_0$; and (b2) $\tau = 2k$.

13.6 DO: (a) Learn about the Platonic solids (tetrahedron, cube, octahedron, dodecahedron, icosahedron). Find the number of vertices, edges, and faces of each. (b) Find the dual of each Platonic solid (viewed as a plane graph drawn on the sphere).

13.7 DO (Dual handshake theorem): Let G be a plane multigraph with r regions; let the i-th region have s_i sides. Prove: $\sum_{i=1}^{r} s_i = 2m$.

13.8 DO: Recall that a "plane" graph means a plane drawing of a graph. (a) Define isomorphism of plane graphs. (b) Find nonisomorphic plane drawings of the same planar graph. (c) Let G be a plane multigraph. Let D be the dual of G. Prove: the dual of D is isomorphic (as a plane graph) to G.

13.9 DO: A *bouquet of m circles* is a multigraph with one vertex and m edges. (Naturally, every edge is a loop.) What is the dual of this multigraph?

13.10 DO: (a) Prove: A plane tree with n vertices has just one region. (b) What is the number of sides of this region? Check your answer against the Dual Handshake Theorem. (c) What is the dual of a plane tree with n vertices?

13.11 DO: The star graph $K_{1,n-1}$ is planar. Count the non-isomorphic plane drawings of this graph.

13.12 DO: Find a 2-connected planar graph that has many non-isomorphic plane drawings.

13.13 CH (Whitney's Theorem, 8 points): Prove: a 3-connected planar graph has a unique plane drawing. (Every pair of plane drawings is isomorphic.) Include your definition of isomorphism of plane drawings.

13.14 DO: (a) Prove Euler's formula for connected plane multigraphs: $n - m + r = 2$. (Review the proof from class.) (b) Modify Euler's formula to make it valid if the graph has c connected components.

13.15 DO: (a) Prove: If G is a connected plane graph with $n \geq 3$ vertices then every region has at least 3 sides. (b) Prove: If G is a planar graph with $n \geq 3$ vertices then $m \leq 3n - 6$. (c) Note that this is false for plane multigraphs. (d) Prove that K_5 is not planar.

13.16 DO: (a) Prove: If G is a triangle-free connected plane graph with $n \geq 3$ vertices then every region has at least 4 sides. (b) Prove: If G is a triangle-free planar graph with $n \geq 3$ vertices then $m \leq 2n - 4$. (c) Prove that $K_{3,3}$ is not planar.

13.17 DO: True or false? "If a planar multigraph G has parallel edges then every plane drawing of G has a 2-sided region."

13.18 DO: (a) Prove: every planar graph has a vertex of degree ≤ 5. (b) Prove: every planar graph is 6-colorable. (Your proof should be very short, a couple of lines.) Remark: The famous "Four-color Theorem" states that every planar graph is 4-colorable.

13.19 DO: (a) Review Kuratowski's Theorem: a graph is planar if and only if it does not contain a Kuratowski subgraph (topological K_5 or topological $K_{3,3}$. Flip through the proof of Kuratowski's Theorem in the Bondy–Murty text. (c) Find a Kuratowski subgraph in the Petersen graph.

13.20 DO: (a) Let $G = (V, E)$ be a graph and $e \in E$ an edge. Prove: If G is planar then G/e is planar. (b) Use part (a) to prove that Petersen's graph is not planar.

13.21 HW (8 points) Prove: the graph G is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor. (See definition in the preamble.) Use Kuratowski's Theorem and Problem 13.20 (a) without proof.

13.22 DO: Find a graph G and and edge e such that G/e has a topological K_5 subgraph but G does not. Your graph should have as few vertices as possible. [This problem erroneously asked the same question about a K_5 as a minor. Error fixed 5-18 11:45pm]

13.23 HW (5+5 points): (a) Give a very simple proof that almost all graphs are not planar. (b) Prove: For all sufficiently large n, the probability that a random graph on n vertices is planar is less than $2^{-0.49n^2}$. (Proving part (b) will earn you partial credit for part (a); for full credit for part (a), you need to give a separate, very simple solution to (a).)

13.24 HW (6+2 points) (a) Prove: If a connected graph G has $m \le n + 2$ edges then G is planar. (b) Show that this becomes false if we drop the condition of connectedness.