Graph Theory – CMSC-27500 – Spring 2015
Instructor: Laszlo Babai
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #14.   Posted 5-27, 9:00pm
Due Thursday, May 28, typeset in **LaTeX**.

Instructor will hold a **problem session** Monday, June 1, 4:30–5:20pm, in Ry-276 (optional, will help you prepare for the next quiz). Come prepared with questions.

Read the homework instructions on the website. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a **separate sheet**, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. *Valency* is a synonym for degree (of a vertex). A graph is *trivalent* if it is regular of degree 3 (every vertex has degree 3). A *bridge* in a graph G is an edge e such that $G - e$ has more connected components than G.

Most problems below were stated in class.

14.1 DO: Solve the Quiz-4 problems (posted) on your time, including the BONUS problems.

14.2 DO: Let H be a graph with n vertices. Count the subgraphs of K_n that are isomorphic to H. (a) Prove: this number is $n!/|\operatorname{Aut}(H)|$ where $\operatorname{Aut}(G)$ denotes the automtorphism group of H (set of self-isomorphisms of H). (b) Verify this formula directly for $H = P_n$ and $H = C_n$.

14.3 DO: **Cayley's formula** says that the number of spanning trees of K_n is n^{n-2}. Verify this formula directly for $n = 6$ and $n = 7$ using the preceding exercise. (List all isomorphism types of trees with 6 and 7 vertices, determine the number of automorphisms of each, and use the formula.)

14.4 DO: Let $n = n_1 + \dots + n_k$ where the n_i are nonnegative integers. (a) Let $A_1, \dots, A_k$ be k symbols. Count the strings of length n that include n_i copies of the symbol A_i for each i. (b) **(Multinomial Theorem)**. The "multinomial coefficient" $\binom{n}{n_1,\dots,n_k}$ is defined as the coefficient of $x_1^{n_1} \dots x_k^{n_k}$ in the expansion of $(x_1 + \dots + x_k)^n$. Prove:

$$\binom{n}{n_1, \dots, n_k} = \frac{n!}{\prod_{i=1}^k n_i!}$$

14.5 DO: Given n and k, determine the number of terms in the multinomial expansion of $(x_1 + \dots + x_k)^n$. Your answer should be a simple closed-form expression (a binomial coefficient).

14.6 DO: Let $d_1, \dots, d_n$ be integers, $d_i \geq 1$ and $\sum_{i=1}^n d_i = 2n-2$. (a) Prove: $(\exists i)(d_i = 1)$. (b) Prove: the number of trees with vertex set $[n]$ and degree $\deg(i) = d_i$ for $i = 1, \dots, n$ is

$$\frac{(n-2)!}{\prod_{i=1}^n (d_i - 1)!}.$$

(Hint: induction on n using (a).)

14.7 DO: Use the preceding exercise to prove Cayley's formula. (Hint: Multinomial Theorem.)

14.8 DO: Study the bijective proof of Cayley's formula using **Prüfer's code.** (Look it up.)

14.9 DO: Prove: for every $r \geq 3$ there exists a graph G with $n = O(r)$ vertices that does not contain K_r and has chromatic number $r + 1$. (Note: the big-Oh notation indicates an *upper bound*.)

14.10 DO: Recall **Dirac's Theorem:** If every vertex of the graph G has degree $\geq n/2$ then G is Hamiltonian. (Hint: saturation method.)

14.11 DO: Recall **Tutte's Theorem:** A graph $G = (V, E)$ has a perfect matching if and only if for every subset $U \subseteq V$ the number of odd components of $G - U$ is at most $|U|$. — Review the proof from class. (Saturation method.)

14.12 HW (8 points; do NOT use print or web sources): **Petersen's Theorem** asserts that every bridgeless trivalent graph has a perfect matching. Derive this result from Tutte's Theorem. (See definitions in preamble.)

14.13 READ: The **4-Color Theorem** (4CT) of Appel and Haken (1989) asserts that every planar graph is 4-colorable. This was a known open problem for about 150 years. — CH: Prove that the 4CT is equivalent to the following statement: Every bridgeless trivalent planar graph is 1-factorable (has edge-chromatic number 3). (The equivalence is Tait's Theorem (1880).) — Note: Tait believed he had also proved that every bridgeless trivalent graph was 1-factorable (thus proving the 4CT). A counterexample was found by Petersen 11 years later.

14.14 DO: Review the proof given in class of **Erdős's Theorem (1959):** For every $g \geq 4$ and every positive $c < 1/(g-1)$, for all sufficiently large n there is a graph G with n vertices, girth $\geq g$, and chromatic number $\geq n^c$.

14.15 DO: review the definition and basic properties of the determinant.

14.16 DO: Compute the determinant of the $n \times n$ matrix

$$\begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{pmatrix}$$

Your answer should be a very simple closed-form expression. Verify your answer by substituting $a = n$ and $b = -1$; you should get $(n+1)^{n-1}$ (Cayley's formula for K_{n+1}).