Graph Theory – CMSC-27500
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #2. Posted 4-2.
Due Tuesday, April 7, 2015 (except where otherwise stated)
Remember: HW problems 1.8 and 1.23 and DO exercises 1.11, 1.12, 1.14, 1.16, 1.22 are also due Tuesday.

Do not submit homework before its due date; it may get lost by the time we need to grade them. If you must submit early, write the early submissions on separate sheets, separately stapled; state "EARLY SUBMISSION" on the top, and send email to the instructor listing the problems you submitted early and the reason of early submission.
Read the homework instructions on the website. The instructions that follow here are only an incomplete summary.

Hand in your solutions to problems marked "HW" and "BONUS." Do not hand in problems marked "DO." Warning: the BONUS problems are underrated. If you hand in solutions to CHALLENGE problems, do so on a separate sheet. PRINT YOUR NAME ON EVERY SHEET you submit. **Request: Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) This will be mandatory after the second week, starting with the solutions you hand in on April 14. Hand in your solutions on paper, do not email.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

In all problems, unless otherwise stated, we have a graph $G = (V, E)$ with n vertices and m edges. The graph $G \setminus v$ is defined by deleting the vertex v from G along with the edges incident with v. In other words, $G \setminus v$ is the induced subgraph of G on the vertex set $V \setminus \{v\}$. – A *tree* is a connected graph with no cycles. – The *girth* of G is the length of its shortest cycle. (If there is no cycle in G then its girth is ∞.) For additional definitions and notation, see also the first HW set.

2.1 DO **(Equivalence classes)** Prove: Every equivalence relation arises from a partition. — More formally, this means the following. Let $\Pi = (B_1, \ldots, B_k)$ be a partition of the set A, i.e., $A = B_1 \cup \cdots \cup B_k$ where $B_i \neq \emptyset$ and $(\forall i \neq j)(B_i \cap B_j = \emptyset)$. The B_i are the *blocks* of this partition. Then Π defines an equivalence relation on A: for $x, y \in A$ we set $x \sim_\Pi y$ if x, y belong to the same block, i.e., $(\exists i)(x, y \in B_i)$.

(Verify that $\sim_\Pi$ is an equivalence relation.) What you need to prove is that if R is an equivalence relation on A then there is a partition Π of A such that $R = \sim_\Pi$. — The blocks of this partition are called the *equivalence classes* of R.

2.2 DO: (a) Prove: congruence modulo m is an equivalence relation on the set of integers. (b) The equivalence classes defined by this relation are called the mod m *residue classes*. Prove that for $m \neq 0$ the number of mod m residue classes is $|m|$.

2.3 DO: (a) Prove: if there is an odd closed walk in G then there is an odd cycle in G. (b) Use this to prove that a graph without odd cycles is bipartite.

2.4 DO: Prove: Every tree with $n \geq 2$ vertices has a vertex of degree 1.

2.5 DO: Prove: If G is a connected graph and v is a vertex of degree 1 then the graph $G \setminus v$ is also connected.

2.6 DO: Use Problems 2.5 and 2.6 to prove that for a tree, $m = n - 1$.

2.7 DO: Prove: a graph is a tree if and only if there is a unique path between each pair of vertices.

2.8 DO: We say that the graph T is a *spanning tree* of the graph G if (a) T is a subgraph of G; (b) $V(T) = V(G)$; and (c) T is a tree. — Prove that G has a spanning tree if and only if G is connected.

2.9 DO: Prove: If G is connected then $m \geq n - 1$.

2.10 DO: Prove: G is a tree if and only if (a) G is connected, and (b) $m = n - 1$.

2.11 DO: Prove: G is a tree if and only if (a) G has no cycles and (b) $m = n - 1$.

2.12 HW (8 points): Draw all nonisomorphic trees with 7 vertices. Clearly state how many you found. (Lose 3 points for each mistake – for missing an isomorphism type and for drawing two isomorphic graphs.)

2.13 HW (6 points): Let T be a tree with n vertices. Determine $P_T(x)$ (the chromatic polynomial of T). Your answer should be a simple closed-form expression (no summation or product symbls, no dot-dot-dots). (Hint: first solve this problem for the case when T is the path of length $n - 1$.)

2.14 DO (due Thursday, April 9): Prove: if G is connected then every pair of longest paths in G intersect (share a vertex).

2.15 BONUS (6 points) Prove: In a tree, all longest paths share a vertex.

2.16 HW (12 points, due Thursday, April 9): Prove: if G is triangle-free then $\chi(G) = O(\sqrt{n})$, i. e., prove that there exists a constant C such that for all sufficiently large n we have $\chi(G) \le C\sqrt{n}$. Explicitly state the smallest constant C for which you can prove this. (Review the big-Oh notation from the instructor's online lecture notes in DM and/or from Rosen.)

2.17 HW (2+2+6 points): Recall that a *matching* in a graph is a set of pairwise disjoint edges. A matching $M \subseteq E$ is *maximal* if no edge can be added to M, i. e., if M is not a proper subset of any matching. A *maximum* matching is a matching of maximum size (largest matching); the *matching number* $\nu(G)$ denotes size of a maximum matching (maximum number of edges in a matching). (a) Determine $\nu(P_n)$ (recall: P_n is the path of length $n-1$) (b) Determine $\nu(C_n)$ (c) Let M be a *maximal* matching in G. Prove: $|M| \ge \nu(G)/2$.

2.18 DO: What is the size of the smallest maximal matching in C_{3k} ? Prove your answer.

2.19 DO: Determine the matching number of (a) the $k \times \ell$ grid (b) the $k \times \ell$ toroidal grid.

2.20 DO: Let G be a regular graph of degree k with girth ≥ 5. (a) Prove: $n \ge k^2 + 1$. (b) Prove: $n = k^2 + 1$ is possible for $k = 1, 2, 3$ (find such graphs). Note: $n = k^2 + 1$ is also possible for $k = 7$ ("Hoffman – Singleton graph"). The Hoffman–Singleton Theorem (1960) says that the only other conceivable value of k for which $n = k^2+1$ could happen is $k = 57$. However, no such graph of degree 57 has been found.

2.21 HW (8 points, due Thursday, April 9): Assume that every vertex of G has degree ≥ 3. Prove that the girth of G is $O(\log n)$.

2.22 BONUS (8 points, due Thursday, April 9) Prove: If G is C_4-free (has no 4-cycles) then $m = O(n^{3/2})$.

2.23 CHALLENGE (8 points, no deadline) Prove that the $O(n^{3/2})$ bound in the preceding exercise is tight, i. e., show that there exists a positive constant c such that for infinitely many values of n (in fact, for all sufficiently large n) there exists a C_4-free graph satisfying $m \ge cn^{3/2}$.