Graph Theory – CMSC-27500 – Spring 2015
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #4. First batch posted 4-9, 8am, updated 10:20am.
Problems 4.16 – 4.31 added at 11:30pm.
Due Tuesday, April 14 (except 4.1 due Apr 9) typeset in **LaTeX**.

Skipping HW set #3 so numbering remains in sync with HW submission dates. "HW set #3" will refer to the items on HW set #2 due Thursday, April 9, viz. HW problems 2.16, 2.21, 2.22 (bonus) and DO exercise 2.14, as well as a review of the Quiz-1 problems as DO exercises.

Do not submit homework before its due date; it may get lost by the time we need to grade them. If you must submit early, write the early submissions on separate sheets, separately stapled; state "EARLY SUBMISSION" on the top, and send email to the instructor listing the problems you submitted early and the reason of early submission.

Read the homework instructions on the website. The instructions that follow here are only an incomplete summary.

Hand in your solutions to problems marked "HW" and "BONUS." Do not hand in problems marked "DO." Warning: the BONUS problems are underrated. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a separate sheet, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. "Iff" is a shorthand for "if and only if." — In this problem sheet, Problems 4.1–4.21 are GRAPH problem, the rest are DIGRAPH problems.

In all GRAPH problems, unless otherwise stated, we have a graph $G = (V, E)$ with n vertices and m edges. — For $u, v \in V$ we write $u \sim_G v$ to indicate that u and v are adjacent in G. We omit the subscript "G" and write $u \sim v$ if the graph in question is clear from the context. — A *forest* is a cycle-free graph. — A set $A \subseteq V$ is a *vertex cover* (or "hittig set") if A intersects ("hits") every edge. The size of the smallest vertex cover is the *covering number* of G, denoted $\tau(G)$ (using the Greek letter "tau"). We simply write τ (omitting "(G)") if the graph in question is clear from the context. (Similarly, we write ν for $\nu(G)$, etc. if there is no risk of confusion.) — For a subset $A \subseteq V$ we write $N_G(A) = \{v \in V \mid (\exists w \in A)(v \sim w)\}$ (the set of neighbors of A). This notation differs from the notation used in class; from now on, this notation will be used. — $\Delta(G)$ denotes the maximum degree of the vertices of G, i. e., $\Delta(G) = \max_{v \in V} \deg(v)$. (In class this quantity was denoted $\deg_{\max}$.) – The **chromatic index** (or *edge-chromatic number*) of a graph is the minimum number of colors needed to color the edges so that adjacent edges have different color. (Two edges are adjacent if they share a vertex.) The chromatic index of G is denoted $\chi'(G)$. According to Vizing's Theorem, $\Delta(G) \leq \chi'(G) \leq 1 + \Delta(G)$. We say that G is of "class 1" if $\chi'(G) = \Delta(G)$ and "class 2" otherwise. — A **1-factor** is a perfect matching. A **1-factorization** of a graph is a partition of E into perfect matchings. (So if a 1-factorization exists then the graph must be regular; and a regular graph admits a 1-factorization iff it is of class 1.) — The vertices of the *line graph* $L(G)$ of a graph $G = (V, E)$ are the edges of G; and vertices e, f of $L(G)$ are adjacent in $L(G)$ if they intersect (as edges of G). — For additional definitions and notation on graphs, see the first two HW sets.

In all DIGRAPH problems, unless otherwise stated, we have a digraph $\vec{G} = (V, E)$ with n vertices and m edges. — For $u, v \in V$ we write $u \to v$ to indicate that there is a directed edge from u to v. — $\deg^+(v)$ is the *out-degree* and $\deg^-(v)$ is the *in-degree* of vertex v. We say that $\vec{G}$ is Eulerian if for every vertex, $\deg^+(v) = \deg^-(v)$. — Walks, paths, closed walks, cycles are defined analogously to the undirected case except now the orientation of the edges must be observed. While in the undirected case the length of a cycle was ≥ 3, for digraphs we permit cycles of length 2 and 1 (loop). — We say that vertex w is *accessible* from vertex v if there exists a $v \to \cdots \to w$ path. — $\vec{G}$ is *strongly connected* if every vertex is accessible from every vertex. — Mutual accessibility is an equivalence relation; its equivalence classes are the *strong components* of $\vec{G}$. So two vertices belong to the same strong component if they are mutually accessible. — A DAG (directed acyclic graph) is a digraph with no cycles. — A *topological sort* of a digraph is a linear order of the vertices such that every edge goes from smaller to larger. — With a digraph $\vec{G}$ we associate an undirected graph $\widetilde{G}$ by ignoring the orientation of the edges and removing loops. So vertices v and w are adjacent in $\widetilde{G}$ if $v \neq w$ and $v \to w$ or $w \to v$ is an edge in $\vec{G}$. We say that G is *weakly connected* if $\widetilde{G}$ is connected. — An *orientation* of a graph $G = (V, E)$ is a digraph $\vec{G} = (V, \vec{E})$ where $\vec{E}$ includes exactly one of $u \to v$ and $v \to u$ for every pair u, v of vertices adjacent in G. So G has 2^m orientations; they have no cycle of length ≤ 2. — A *tournament* is an orientation of the complete graph.

GRAPH PROBLEMS

4.1 DO, due Thursday, April 9 **(First Quiz)** Review and solve the problems of the first quiz (posted), including the Bonus problem, before Thursday's class.

4.2 DO: Let $c(G)$ denote the number of connected components of G. (a) Prove: G is a forest if and only if every connected component of G is a tree. (b) Prove: if G is a forest then $c(G) = n - m$. (c) Prove that for every graph G we have $c(G) \geq n - m$. (d) Prove: G is a forest if and only if $c(G) = n - m$.

4.3 DO: (a) Prove: a set $A \subseteq V$ is a vertex cover iff its complement $V \setminus A$ is an independent set. (b) Prove: $\tau = n - \alpha$. (c) Determine ν and τ for paths, cycles, complete graphs, and complete bipartite graphs. (d) Determine ν and τ for grids and for toroidal grids. (e) Determine all graphs for which $\tau = 1$. (f) Determine all graphs for which $\nu = 1$.

4.4 DO: Review the König–Hall "Marriage Theorem:" Let G be a bipartite graph with vertex partition $V = L \cup R$ (all edges connect L and R). We say that L is "fully matched" if there exists a matching of size $|L|$. A subset $A \subseteq L$ is a *König–Hall obstacle* if $|N_G(A)| < |A|$. Theorem: L is fully matched iff there is no König–Hall obstacle, i. e., iff $(\forall A \subseteq L)($ the König–Hall condition $|N_G(A)| \geq |A|$ holds $)$. [Typo in last line corrected 4-13 4pm]

4.5 DO: (a) Prove: $\nu \leq \tau$ for all graphs. (b) Find a graph G such that $\nu(G) < \tau(G)$. (c) Prove: $\tau \leq 2\nu$ for all graphs. (d) For every $k \geq 0$ find a graph G such that $\nu(G) = k$ and $\tau(G) = 2k$. Let G have as few edges as possible. (e) For every $k \geq 0$ find a **connected** graph G such that $\nu(G) = k$ and $\tau(G) = 2k$.

4.6 DO: (a) Review König's Theorem: If G is bipartite then $\nu(G) = \tau(G)$. (b) Deduce the Marriage Theorem from König's Theorem. (c) Find a non-bipartite graph G for which $\nu(G) = \tau(G)$.

4.7 DO: Let A be a matrix. A $k \times \ell$ submatrix is obtained by selecting k rows and ℓ columns and deleting every row and every column not selected. Count the $k \times \ell$ submatrices of an $n \times m$ matrix.

4.8 DO: Let us say that a $k \times \ell$ submatrix of an $n \times n$ matrix is *fat* if $k+\ell \geq n+1$. (This is local terminology, i. e., it has been invented just for this problem.) Let X be the $n \times n$ matrix of whose (i, j) entry is the variable x_{ij}. Let B be a matrix obtained from X by replacing some of the x_{ij} by zero. So each entry of B is either zero or a variable, and all variables occurring are distinct. Consider the determinant $\det(B)$; this is a multivariate polynomial which is linear (of degree ≤ 1) in each variable. Prove: $\det(B) = 0$ (the identically zero polynomial) iff B has a fat submatrix consisting only of zeros.

4.9 DO: Let $M \subseteq E$ be a matching and $C \subseteq V$ a cover of a (not necessarily bipartite) graph G. Prove: If $|M| = |C|$ then both of them are optimal, i. e., $\nu(G) = |M| = |C| = \tau(G)$.

4.10 DO: (a) Review the proof of König's Theorem (alternating paths, augmenting paths, finding a matching and a cover as in the preceding exercise). (b) (Optional) The proof provides an efficient algorithm to find a maximum matching and a minimum cover in a bipartite graph. Estimate the running time of the algorithm in terms of n and m.

4.11 DO: (a) Prove: The chromatic index of G is $\geq \Delta(G)$ (the maximum degree of G). (See def. in the "Definitions, notation" section before the exercises.) (b) Understand **Vizing's Theorem:** $\chi'(G)$, the chromatic index of G, is either Δ or $1 + \Delta$. — Graphs for which $\chi'(G) = \Delta(G)$ are called "class-1 graphs," all others "class 2." (c) Prove: If G is regular of degree k then G is of class 1 iff E is the union of k perfect matchings. (These perfect matching are then necessarily disjoint.) — A perfect matching is also called a **1-factor** (a subgraph in which every vertex has degree 1; d-factors are defined analogously). A decomposition of E into perfect matchings is called a **1-factorization**. So a regular graph is of class 1 iff it has a 1-factorization. (d) Prove: If G is k-regular of class 1 and $k \geq 1$ then n is even.

4.12 DO: (a) Let G be a *trivalent graph*, i. e., a regular graph of degree 3. Prove: if G is Hamiltonian then G is of class 1. (b) Prove: Petersen's graph is of class 2. Note that it follows from this statement that Petersen's graph is not Hamiltonian.

4.13 DO: (a) Prove: If G is a non-empty regular bipartite graph then G has a perfect matching (i. e., $\nu(G) = n/2$). ("Non-empty" means $m > 0$.)

(b) Prove: Every regular bipartite graph is of class 1 (i. e., it has a 1-factorization).

4.14 DO: **(Scheduling a round-robin tournament)** Prove: For every even number n, a round-robin chess tournament of n players can be scheduled in $n-1$ rounds (so each player plays in every round). (In a round-robin tournament, each player plays against each player exactly once.) In other words, prove that K_n is of class 1 (for even n). (Hint: there is a very simple explicit 1-factorization of K_n, using regular polygons.) — Comment: the start of a chess tournament in the 19th century was delayed because the judge forgot to bring his 1-factorization table along.

4.15 DO: (a) Study the instructor's "Puzzle problem" collection (accessible from his REU page). (b) Solve Problem 5 (dominoes) on the Puzzle sheet. (c) Rephrase the domino problem as a question about the existence of a perfect matching in a graph. (d) Solve Problem 6 (triominoes) on the Puzzle sheet. (e) CHALLENGE (6 points): Solve Problem 7 (band-aids) on the Puzzle sheet.

4.16 HW (10 points) Beyond the seven seas there is a tiny island, 6 square miles in all. The island is inhabited by six native tribes and by six turtle species. Each tribe and each turtle species occupies one square mile of territory; the territories of the tribes don't overlap with one another; nor do the territories of the different turtle species.

Each tribe wishes to select a totem animal from among the turtle species found in the tribe's territory; and each tribe must have a different totem animal.

Prove that such a selection is always possible. You may use, without proof, a DO exercise stated in this problem set.

4.17 HW (8+8 points) As before, $P_G(x)$ denotes the chromatic polynomial of the graph G. Recall that $P_{P_n}(x) = x(x-1)^{n-1}$ (where P_n is the path of length $n-1$).
(a) Prove: for $n \geq 4$ we have

$$P_{C_n}(x) = P_{P_n}(x) - P_{C_{n-1}}(x).$$

(b) Use part (a) to prove that

$$P_{C_n}(x) = (-1)^n \left((1-x)^n - (1-x)\right).$$

4.18 HW (7 points) Prove: the line graph $L(K_{r,s})$ of the complete bipartite graph $K_{r,s}$ is the Cartesian product of two graphs. Which two graphs?

4.19 DO (a) How is $L(K_5)$ related to the Petersen graph? (b) Use this connection to prove that the Petersen graph has 120 automorphism. (c) (Optional) Prove that the automorphism group of the Petersen graph is isomorphic to S_5, the symmetric group of degree 5.

4.20 DO Prove: $\chi(L(G)) = \chi'(G)$. (The chromatic number of the line graph is the chromatic index of the graph.)

4.21 DO Find the smallest graph G with the following properties: G is connected and contains a cycle C such that the removal of any vertex of C makes the graph disconnected. ("Smallest" means fewest edges.)

DIGRAPH PROBLEMS

4.22 DO **(Directed Handshake Theorem)**:

$$\sum_{v\in V} \deg^+(v) = m = \sum_{v\in V} \deg^-(v).$$

4.23 DO: (a) Prove: If there exists a walk from u to v then there is a path from u to v. (b) Prove: accessibility is a reflexive and transitive relation.

4.24 DO: (a) Prove: mutual accessibility is an equivalence relation. — Its equivalence classes are called the *strong components* of the digraph. (b) Prove: $\vec{G}$ is strongly connected iff there is just one strong component. (c) What are the strong components of a DAG?

4.25 HW (4+12 points): (a) Draw a weakly but not strongly connected digraph in which every vertex belongs to a cycle of length ≥ 3. Make your digraph have as few edges as possible. (b) Prove: If an Eulerian digraph is weakly connected then it is strongly connected.

4.26 DO: Prove: a digraph admits a topological sort iff it is a DAG.

4.27 HW (6 points): [updated Apr 12 3:20am: loop-free condition added] We say that a digraph is *loop-free* if it has no loops (cycles of length 1). Prove: "half of every loop-free digraph is a DAG." More precisely, let $\vec{G} = (V, E)$ be a loop-free digraph. Prove that there exists a subset $F \subseteq E$ such that $|F| \geq |E|/2$ and (V, F) is a DAG. (Hint: your proof should be no longer than two lines.) **Do NOT collaborate on this problem.** It is an "Ah-ha" problem, you cannot collaborate without hearing or revealing the complete solution. Think!

4.28 DO: Prove: Every tournament has a Hamilton path.

4.29 DO: Prove: Every strongly connected tournament with $n \geq 3$ vertices is Hamiltonian (has a Hamilton cycle).

4.30 DO: We shall prove **Stanley's Theorem:** The number of acyclic orientations of a graph G is $(-1)^n P_G(-1)$ (where $P_G(x)$ is the chromatic polynomial of G). Verify this statement for the (a) complete graphs (b) trees (c) cycles.

4.31 CHALLENGE (6 points) Prove: for every $n \geq 3$ there exists a tournament with n vertices that contains at least $(n-1)!/2^n$ Hamilton cycles. [Formula corrected 4-14 4pm]