Graph Theory – CMSC-27500 – Spring 2015
http://people.cs.uchicago.edu/~laci/15graphs
Homework set #5. Posted 4-14, 5:30pm, final update 11:45pm
Due Thursday, April 16, typeset in **LaTeX**.

Do not submit homework before its due date; it may get lost by the time we need to grade them. If you must submit early, write the early submissions on separate sheets, separately stapled; state "EARLY SUBMISSION" on the top, and send email to the instructor listing the problems you submitted early and the reason of early submission.

Read the homework instructions on the website. The instructions that follow here are only an incomplete summary.

Hand in your solutions to problems marked "HW" and "BONUS." Do not hand in problems marked "DO." Warning: the BONUS problems are underrated. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a separate sheet, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. Notation: for a non-negative integer n we write $[n] = \{1, \ldots, n\}$. So $[0] = \emptyset$, $[1] = \{1\}$, $[2] = \{1, 2\}$, $[3] = \{1, 2, 3\}$, etc.

"LN" refers to the instructor's online Discrete Mathematics Lecture Notes. Read LN Chapter 7 ("Finite Probability Spaces") for the relevant definitions and basic facts. This preamble and the problem set describe only some of these basics.

Let Ω be a non-empty finite set. A *probability distribution* over Ω is a function $P : \Omega \to \mathbb{R}$ such that

$$(\forall x \in \Omega)(P(x) \geq 0) \text{ and } \sum_{x \in \Omega} P(x) = 1.$$

The *uniform distribution* is defined by setting $(\forall x \in \Omega)(P(x) = 1/|\Omega|)$.

A *finite probability space* is a pair (Ω, P), where Ω is a non-empty finite set and P is a probability distribution over Ω. If only Ω is specified, we assume P is uniform; this will often but not always be the case in applications to graph theory.

We call Ω the **sample space** and think of it as the set of all possible outcomes of an experiment (such as a shuffled deck of cards ($|\Omega| = 52!$), a poker hand ($|\Omega| = \binom{52}{5}$)), or a sequence of n coin flips ($|\Omega| = 2^n$)). The elements of Ω are called *elementary events.* The *events* are the subsets of Ω. The probability of the event $A \subseteq \Omega$ is defined as $P(A) = \sum_{x\in\Omega} P(x)$. Under uniform distribution we have $P(A) = |A|/|\Omega|$ for all events A ("naive probability"). The event A is *trivial* if $P(A) = 0$ or $P(A) = 1$.

Let B be an event such that $P(A) \neq 0$. The *conditional probability* $P(A \mid B)$ (the probability of A given B) is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

A *Boolean formula* is a formula obtained by repeatedly applying the Boolean operations of union, intersection, and complementation to variables interpreted as sets, e. g., $f(A, B, C, D) = (A \cap D) \cup \overline{((A \cap B) \cup C) \cap (\overline{B} \cup D)}$.

Events A and B are *independent* if $P(A \cap B) = P(A)P(B)$; they are *positively correlated* if $P(A \cap B) > P(A)P(B)$; and *negatively correlated* if $P(A \cap B) < P(A)P(B)$. Events $A_1, \ldots, A_k$ are *independent* if $\forall I \subseteq [k]$

$$P\left(\bigcap_{i\in I} A_i\right) = \prod_{i\in I} P(A_i).$$

Other terms used to express independence are "fully independent" and "mutually independent"; they just mean the same as "independent." We sometimes add "fully" before "independent" to emphasize the distiction from pairwise independence.

A *random variable* over the probability space (Ω, P) is a function $X : \Omega \to \mathbb{R}$. The *expected value* of X is $E(X) = \sum_{x\in\Omega} X(x)P(x)$. For an event $A \subseteq \Omega$, the *indicator* of $A \subseteq \Omega$ is the random variable $I_A : \Omega \to \{0, 1\}$ defined by setting

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \Omega \setminus A. \end{cases}$$

A *Bernoulli trial* is an experiment with two possible outcomes called "success" and "failure." Let p denote the probability of success; so we can think of the Bernoulli trial as flipping a biased coin which comes up Heads ("success") with probability p. A "sequence of Bernoulli trials" refers to independent repetition of a Bernoulli trial, i. e., a sequence of independent events $A_1, \ldots, A_n$ where A_i represents the event of success of the i-th trial (the i-th coin came up Heads); $P(A_i) = p$.

The problems in this sheet will always refer to a fixed finite probability space (Ω, P) (except for 5.24).

5.1 DO: Study the cards of the standard deck (52 cards divided into four suits of 13 kinds ("ranks")) and the various "hand strength" (card combinations in a hand of five cards) in poker (such as "1 pair", "2 pair", "3 of a kind", "straight", "flush", "full house", "4 of a kind", "straight flush", "royal flush"). Calculate the probability of each hand strength when a hand of 5 cards is randomly dealt.

5.2 DO: Study the **"Finite probability spaces"** chapter from LN

5.3 DO: (a) $P(\emptyset) = 0$ and $P(\Omega) = 1$. (b) Let $\overline{B}$ denote the complement of the event B, i.e., $\overline{B} = \Omega \setminus B$. Prove: $P(\overline{B}) = 1 - P(B)$.

5.4 DO (modular identity): Let A, B be events (over the same probability space). Prove: $P(A \cup B) + P(A \cap B) = P(A) + P(B)$.

5.5 DO (additivity): We say the events A and B are "almost disjoint" if $P(A \cap B) = 0$. (If A and B are disjoint then they are almost disjoint but not necessarily conversely.) Let $A_1, \ldots, A_k$ be pairwise almost disjoint events. Prove: $P(\bigcup_{i=1}^k A_i) = \sum_{i=1}^k P(A_i)$.

5.6 DO (union bound): Let $A_1, \ldots, A_k$ be events. Prove:

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i).$$

5.7 DO: (a) Prove: If A is a trivial event then for any event B, the events A and B are independent. (b) Prove: If A is a trivial event tand $B_1, \ldots, B_k$ are independent events then $A, B_1, \ldots, B_k$ are independent. (c) Prove: A and A are independent iff A is a trivial event. (d) Prove: If A, B are independent events then A and $\overline{B}$ are also independent. (e) Infer from this that $\overline{A}$ and $\overline{B}$ are also independent (without repeating the argument). (f) Assume $A_1, \ldots, A_k$ are independent events. For each i, let C_i be either A_i or $\overline{A}_i$. Prove that $C_1, \ldots, C_k$ are independent. (g) Assume A, B, C, D are independent events. Prove that $A \cup B$, C, D are also independent. (h) **(Boolean combinations of disjoint sets of independent events are independent)** Assume $A_1, \ldots, A_k$ are independent events. Let $\Pi = (R_1, \ldots, R_t)$ be a partition of the set $[k] = \{1, \ldots, k\}$. Let f_i be a Boolean formula in

$|R_i|$ variables ($i = 1, \ldots, t$) and let $B_i = f_i(A_j : j \in R_i)$ (the Boolean formula f_i applied to the events corresponding to the i-th block of the partition). Then $B_1, \ldots, B_t$ are independent.

5.8 DO: Prove: If there exist k non-trivial independent events in a probability space of size n then $n \geq 2^k$. (The size of the probability space (Ω, P) means the size of the sample space, $|\Omega|$.)

5.9 HW (6 points): Find a probability space and three events that are pairwise but not fully independent. Make your probability space as small as possible. Prove that it is the smallest with reference to a "DO" exercise on this sheet; you do not need to prove that exercise.

5.10 BONUS (5 points): Find a probability space and k events that are $(k-1)$-wise but not fully independent. Make your probability space as small as possible. Prove that it is the smallest with reference to a "DO" exercise on this sheet; you do not need to prove that exercise. A correct solution to this exercise also earns you the 6 points for the preceding problem. Full clarity and simplicity are paramount; complicated solutions get partial credit even if fully correct.

5.11 DO: Consider the uniform probability space over a sample space Ω where $|\Omega| = p$ is a prime number. Prove: if A, B are non-trivial events then they are not independent.

5.12 CHALLENGE (5+5 points) (a) Construct a probability space of size $O(k)$ with k pairwise independent non-trivial events. (b) Do the same with triple-wise independent events.

5.13 CHALLENGE (6+4 points) Prove: (a) If there exist k pairwise independent non-trivial events in a probability space then the size of the space is $\geq k+1$. (b) If there exist k four-wise independent non-trivial events in a probability space then the size of the space is $\Omega(k^2)$ (i. e., it is $\geq ck^2$ for some constant $c > 0$ and all sufficiently large k).

5.14 DO: LN 7.1.8 and 7.1.14 (independence/correlation of "sum of dice" events)

5.15 DO: (a) Let B be an event such that $P(B) \neq 0$. For $x \in B$ let $P_B(x) = P(x)/P(B)$. Prove: (B, P_B) is a probability space and for all $A \subseteq B$ we have $P_B(A) = P(A \mid B)$. (b) Let C be a nontrivial event. For an event $A \subseteq \Omega$, let $(A \mid C)$ ("A given C") denote the event $A \cap C$ in the probability space (C, P_C). (b1) Construct a probability space and three events A, B, C such that A and B are not independent but $(A \mid C)$ and $(B \mid C)$ are independent and $(A \mid \overline{C})$ and $(B \mid \overline{C})$ are also independent. (b2) Same as (b1) but make all the events mentioned, including $(A \mid C)$, etc., nontrivial.

5.16 DO: LN 7.1.9 (Theorem of Complete Probability)

5.17 DO (Probability of causes): Diseases A and B have similar symptoms. Let W be the population of all patients showing these symptoms. The two diseases can only be differentiated by costly tests. We know (from sampling the population and performing these costly tests) that 70% of W have disease A, 25% have disease B, and 5% have some other disease. We consider the effectiveness of treatment T. We know that 60% of the patients with disease A respond to T, while only 12% of the patients with disease B respond to treatment T. From the rest of the population W, 40% respond to treatment T.

(a) A new patient arrives at the doctor's office. The doctor determines that the patient belongs to W. What is the probability that the patient will respond to treatment T?

(b) The patient's insurance will not pay for the expensive tests to differentiate between the possible causes of the symptoms. The doctor bets on treatment T. A week later it is found that the patient did respond to the treatment. What is the probability that the patient had disease A? Show all the intermediate results you need to compute.

5.18 DO: Let X be a random variable. Recall the definition of the expected value $E(X)$ (see preamble of this problem sheet). Prove:

$$\min X \leq E(X) \leq \max X.$$

5.19 DO: (a) Let X be a random variable. Prove:

$$E(X) = \sum_{y \in \mathbb{R}} y P(X = y).$$

(This seemingly infinite sum over all real numbers is in fact just a sum over the range of X, a finite and often very small set; in part (b) below it is just $\{0, 1\}$.)
(b) Let I_A denote the indicator of A. Prove: $E(I_A) = P(A)$.

5.20 DO (linearity of expectation): (a) (abridged) Let X, Y be random variables (over the same probability space). Prove: $E(X+Y) = E(X) + E(Y)$. (b) (unabridged) Let $X_1, \ldots, X_k$ be random variables (over the same probability space) and let the α_i be real numbers. Prove:

$$E\left(\sum \alpha_i X_i\right) = \sum \alpha_i E(X_i).$$

5.21 DO: We repeat n times a Bernoulli trial with probability p of success (flip a biased coin n times; the coin comes up Heads with probability p and Tails with probability $1-p$). Let X denote the number successes (heads). Prove: $E(X) = pn$. (Note: The sample space in this problem has size 2^n.)

5.22 HW (9+2 points): (a) LN 7.2.13 (club with 2000 members). Assume the club serves vodka legally to all its members. - Make sure you give a clear definition of the random variables you use. The clarity of the definition accounts for 2/3 of the credit. Explain the role of the vodka. Hint. Learn about indicator variables (LN 7.2.6, 7.2.7). Represent the number of lucky members as a sum of indicator variables. (b) State the size of the sample space for this experiment.

5.23 DO: Consider a *random tournament* T on a given set V of n vertices. (Flip an unbiased coin for each edge of the complete graph to decide its orientation.) (a) Observe that the sample space for this experiment has size $2^{\binom{n}{2}}$. (b) Let X denote the number of Hamilton cycles in T. Prove: $E(X) = (n-1)!/2^n$. (c) Infer from this that there exists a tournament with at least $(n-1)!/2^n$ Hamilton cycles (Szele's Theorem). (This is a lot of Hamilton cycles!). — Note that we have proved the **existence** of such a tournament without being able to construct one. This method of proving the existence of an object is called the **probabilistic method**.

5.24 DO (due Tuesday, April 20): Let us consider a Bernoulli trial with probability p of success. Repeat the trial until the first success. Let X be the number of trials performed. Prove: $E(X) = 1/p$. — Part of your problem is to clarify what the question means. This is not a finite probability space; the expected value in question will be an infinite sum.