Graph Theory – CMSC-27500 – Spring 2015
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #6. Posted 4-17, 2am, updated 6:30am
Problem 6.20 added at 4:40pm, 6.21 at 11:45pm.
Due Tuesday, April 21, typeset in **LaTeX**.

Do not submit homework before its due date; it may get lost by the time we need to grade them. If you must submit early, write the early submissions on separate sheets, separately stapled; state "EARLY SUBMISSION" on the top, and send email to the instructor listing the problems you submitted early and the reason of early submission.

Read the homework instructions on the website. The instructions that follow here are only an incomplete summary.

Hand in your solutions to problems marked "HW" and "BONUS." Do not hand in problems marked "DO." Warning: the BONUS problems are underrated. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a separate sheet, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. As before $G = (V, E)$ denotes a graph with n vertices and m edges.

Notation: for a non-negative integer n we write $[n] = \{1, \ldots, n\}$. So $[0] = \emptyset$, $[1] = \{1\}$, $[2] = \{1, 2\}$, $[3] = \{1, 2, 3\}$, etc.

Let $G = (V, E)$ be a graph and $e = \{u, v\} \in E$ an edge. The graph $G - e$ is defined as $G - e = (V, E \setminus \{e\})$ **(deletion of edge e)**; note that no vertex has been deleted. (In class I used the notation $G \setminus e$ for this; from now on, I will use $G - e$ which is the more commonly used notation.) The graph G/e is defined as $G/e = (V', E')$ where V' is obtained from V by identifying u and v (so $|V'| = |V| - 1$); let us call the new vertex w. Adjacency is G/e is defined as follows. Let $x, y \in V'$. If $x \neq w$ and $y \neq w$ then $x \sim_{G/e} y$ iff $x \sim_G y$. If $y = w$ then $x \sim_{G/e} y$ iff $x \sim_G u$ or $x \sim_G v$ **(contraction of e)**. So for instance $K_n/e = K_{n-1}$ for all n and $C_n/e = C_{n-1}$ for $n \geq 4$.

"LN" refers to the instructor's online Discrete Mathematics Lecture Notes. Read LN Chapter 7 ("Finite Probability Spaces") for the relevant definitions and basic facts. This preamble and the problem set describe only some of these basics.

Chromatic number of digraphs. A legal colorig for a digraph is defined analogously to a legal coloring in a graph: For a coloring $f : V \to \{\,\text{colors}\,\}$ to be legal, we need that if $u \to v$ is an edge then $f(u) \neq f(v)$. So for instance a digraph with a loop has no legal coloring. For a loop-free digraph G, the chromatic number $\chi(G)$ is the minimum number of colors needed for a legal coloring. This is the same as $\chi(\widetilde{G})$ where $\widetilde{G}$ is the undirected graph obtained from G by ignoring orientations.

6.1 DO: Review the exercises from the previous problem sheets. Remember: the next quiz is Tuesday, April 21. It will contribute 8% to your course grade.

6.2 DO: Prove the deletion/contraction recurrence for the chromatic polynomial: If $G = (V, E)$ and $e \in E$ then

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x).$$

6.3 DO: Let $\mathrm{DAG}(G)$ denote the number of acyclic orientations of the graph G. (Warning: this is local notation.) Prove the following deletion/contraction recurrence for this quantity:

$$\mathrm{DAG}(G) = \mathrm{DAG}(G - e) + \mathrm{DAG}(G/e).$$

6.4 DO: Prove **Richard Stanley's Theorem** (1972), an unexpected connection between colorings and acyclic orientations:

$$\mathrm{DAG}(G) = (-1)^n P_G(-1).$$

Hint. By induction on m, using the two preceding exercises. The base cases are the graphs with $m = 0$, i. e., $G = \overline{K}_n$.

6.5 DO: **(exponential decay beats polynomial growth)** Let $c > 0$ and k be constants. Assume $f(n) = O(n^k)$ and $g(n) \geq 2^{cn}$. Prove: $\lim_{n\to\infty} f(n)/g(n) = 0$.

6.6 DO: Prove: $\binom{n}{k} \leq n^k/k!$.

6.7 DO: Prove: $\mathrm{e}^x \geq 1 + x$ for all $x \in \mathbb{R}$.

6.8 DO: Let $0 < x < 1$ and let k be a positive integer. Prove: $(1 - x)^k \geq 1 - kx$.

6.9 DO: Let $T = (V, E)$ be a tournament. We say that T is k*-paradoxical* if it has $n \geq k+1$ vertices and for every set $A \subseteq V$ of $|A| = k$ vertices there is a vertex $x \in V \setminus A$ such that $x \to A$ (x *dominates* A, i. e., $(\forall u \in A)(x \to u)$). Prove (recall from class): if

$$\binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} < 1 \tag{1}$$

then there exists a k-paradoxical tournament with n vertices.

6.10 DO (Erdős): Fix k. Prove: almost all tournaments are k-paradoxical. — Explanation. Consider a random tournament T on a given set of n vertices. (The orientation of each edge is decided by flipping a coin.) Let p_n denote the probability that T is k-paradoxical. Then $\lim_{n\to\infty} p_n = 1$.

6.11 DO: Prove: there exists a constant C such that for all k, if $n > Ck^2 2^k$ then there exists a k-paradoxical tournament with n vertices. – Note: k is not a constant in this exercise and C must not depend on k. Hint: Verify that Equation (1) in Problem 6.9 holds. Use 6.6 and 6.7.

6.12 HW (16 points): Prove: almost all graphs have diameter 2. — Explanation: Consider a random graph G on a given set of n vertices; adjacency is decided by flipping a coin for each pair of vertices. Let p_n denote the probability that G has diameter 2. Prove: $\lim_{n\to\infty} p_n = 1$. (Recall that the distance $\text{dist}(u, v)$ between vertices u, v is the length of the shortest $u - \cdots - v$ path; and the diameter of G is $\text{diam}(G) = \max_{u,v\in V} \text{dist}(u, v)$.)

6.13 DO: (a) Prove: almost all tournaments are not DAGs. (b) Prove: for all sufficiently large n, the probability that a random tournament on n vertices is a DAG is less than $2^{-0.49n^2}$.

6.14 DO: Assume $n \equiv 0$ or $1 \pmod 4$. (a) Prove: almost all graphs are not self-complementary. (b) Prove: for all sufficiently large n, the probability that a random graph on n vertices is self-complementary is less than $2^{-0.24n^2}$ and more than $2^{-0.25n^2}$.

6.15 DO (due Thursday, April 23; the parts involving automorphisms are optional): **(Paley tournament)** Let p be a prime, $p \equiv -1 \pmod 4$. We define the digraph Pa_p as follows: $V(\text{Pa}_p) = \{0, \ldots, p-1\}$ and for $u, v \in V(\text{Pa}_p)$ we have the edge $u \to v$ if $u \neq v$ and $v - u$ is a quadratic residue mod p, i. e., $(\exists x \in \mathbb{Z})(v - u \equiv x^2 \pmod{p})$. (In class I denoted this tournament by P_p, but that notation has already been reserved for the path of length $p-1$.) (a) Prove that this is a tournament. Indicate where you use the condition that $p \equiv -1 \pmod 4$. (b) Prove that the automorphism group of Pa_p is *edge-transitive*, i. e., for every pair of edges, e, f, there is an automorphism (self-isomorphism) of Pa_p that takes e to f. (c) Prove that Pa_p is *self-converse*, i. e., if we reverse every edge, the tournament obtained is isomorphic to Pa_p. (d) If instead we take $p \equiv 1 \pmod 4$ then the same definition gives a graph (the **Paley graph**, which we also denote Pa_p). (e) Prove that the automorphism group of the Paley graph is *arc-transitive*, i. e., for every pairs of adjacent pairs of vertices, (u, v) and (u', v'), there is an automorphism that takes u to u' and v to v' (f) Prove that the Paley graph is self-complementary. — Later we shall see that all sufficiently large Paley tournaments are k-paradoxical (Graham – Spencer Theorem). The proof is based on "Weil's character sum estimate," a deep result in number theory.

6.16 DO: **(Erdős–Rényi random graphs)** Let $0 < p < 1$. Let V be a given set of n vertices. Let us construct a random graph G on vertex set V using a biased coin with probability p of coming up Heads to decide adjacency among pairs of vertices ("Heads" means "adjacent"). This procedure defines the probability space $\mathbf{G}_{n,p}$: the sample space is the set of all graphs on vertex set V (so it has size $2^{\binom{n}{2}}$); if H is a graph with the given set of vertices and m edges then $P(G = H) = p^m(1-p)^{\binom{n}{2}-m}$. (Verify!) (a) What is the expected number of edges in G? (b) What is the expected number of triangles in G? (c) What is the expected number cycles of length k in G? — Your answers should be very simple expressions.

6.17 DO: Prove: "half of every graph is bipartite." — Explanation: Given a graph $G = (V, E)$, prove that there is a subset $F \subseteq E$ such that $|F| \geq |E|/2$ and (V, F) is bipartite. — Give two proofs: (a) a very elegant 3-line proof using the probabilistic method; (b) by a very simple deterministic algorithm. (You need to prove that the algorithm achieves the desired goal.)

6.18 CHALLENGE (16 points): Let $k \geq 3$ and $0 < \delta < 1/k$. Prove: for all sufficiently large n there is a graph with n vertices, $\geq n^{1+\delta}$ edges, and girth greater than k. Hint. Use the Erdős–Rényi model. You may use the Chernoff bound (see LN, last section of the chapter on Finite Probability Spaces).

6.19 CHALLENGE (12 points): Fix $\epsilon > 0$. Prove: for all sufficiently large n, if G is a graph with n vertices and $m \geq n(\log n)^2$ edges then G has an orientation $\vec{G}$ such that every sub-DAG of $\vec{G}$ has at most $(m/2)(1+\epsilon)$ edges.

6.20 HW (6+4 points, due Thursday, April 23): (a) Let G be a loop-free digraph. Assume every vertex in G has out-degree $\leq k$. Prove: $\chi(G) \leq 2k+1$. (b) Prove that this bound is tight: for every k, construct a loop-free digraph G such that the out-degree of every vertex is $\leq k$ and $\chi(G) = 2k+1$. [Points for part (b) added 4-29]

6.21 DO: Let $k \geq 1$. Prove: a graph G is k-colorable iff G has an acyclic orientation $\vec{G}$ such that $\vec{G}$ has no path of length k. [Typo corrected 4-21 12:15am]