Graph Theory – CMSC-27500 – Spring 2015
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #8. Posted 4-25, 11:20am
Due Tuesday, April 28, typeset in **LaTeX**.

Do not submit homework before its due date; it may get lost by the time we need to grade them. If you must submit early, write the early submissions on separate sheets, separately stapled; state "EARLY SUBMISSION" on the top, and send email to the instructor listing the problems you submitted early and the reason of early submission.

Read the homework instructions on the website. The instructions that follow here are only an incomplete summary.

Hand in your solutions to problems marked "HW" and "BONUS." Do not hand in problems marked "DO." Warning: the BONUS problems are underrated. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a **separate sheet**, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. As before $G = (V, E)$ denotes a graph with n vertices and m edges. $\omega(G)$ denotes the clique number of G, i. e., the number of vertices of the largest clique (complete subgraph) of G. So for instance $\omega(G) \leq 2$ means G is triangle-free.

This homework set concerns the elements of **Ramsey Theory**, started in a paper by Paul Erdős and György Szekeres written while they were undergraduates in Budapest in the early 1930s and subsequently developed by Erdős and Richard Rado. The Erdős–Rado arrow symbol $n \to (k, \ell)$ means the following statement: "If we color the edges of K_n red and blue then either there is a red clique of size k or there is a blue clique of size ℓ." Correspondingly, $n \not\to (k, \ell)$ means there exists a red/blue coloring of the edges of K_n such that there is neither a red K_k nor a blue K_ℓ. A special case of Ramsey's Theorem states that $(\forall k, \ell)(\exists n)(n \to (k, \ell))$. Philosopher, mathematician, and economist Frank Plumpton Ramsey (1903–1930) published a much more general result in 1928 in a paper titled "On a problem of formal logic."

8.0 DO: F. P. Ramsey died at the age of 26. He made important contributions to philosophy, economic theory, and mathematical logic. Study his biography.

8.1 DO: Observe: $\omega(G) = \alpha(\overline{G})$.

8.2 HW (3 points): Prove that the Erdős–Rado relation $n \to (k, \ell)$ is equivalent to the following statement: "For every graph G on n vertices, either $\alpha(G) \geq k$ or $\omega(G) \geq \ell$." – Your solution should be one line, giving the correspondence between the red/blue coloring and the graph G. Beyond an accurate and clear description of this correspondence (in each direction), no further proof is needed.

8.3 DO: Review the proofs of the relations $6 \to (3,3)$, $5 \not\to (3,3)$, $10 \to (4,3)$, $k \to (k,2)$, discussed in class. Rephrase each proof in terms of $\alpha(G)$ and $\omega(G)$ as in the preceding exercise.

8.4 HW (8 points): Prove: $9 \to (4,3)$. Hint: Follow the proof of the relation $10 \to (4,3)$ discussed in class, reinterpreted in terms of $\alpha(G)$ and $\omega(G)$ as in the preceding exercise. Characterize the degree of each vertex of the graphs G for which the proof breaks down; then show that such graphs G do not exist.

8.5 HW (6 points): Prove that $8 \not\to (4,3)$. Use the wording from Problem 8.2. So what you need to do is find a graph G with 8 vertices such that $\alpha(G) \leq 3$ and $\omega(G) \leq 2$. (The latter means G is triangle-free.) Your solution should be a nice symmetrical drawing (by hand) of such a graph G. Do not prove.

8.6 DO (Erdős–Szekeres Theorem): Prove that for $k, \ell \geq 1$ the following relation holds:

$$\binom{k+\ell}{k} \to (k+1, \ell+1).$$

Hint: Induction on $k + \ell$. The bases cases are those when $k = 1$ or $\ell = 1$ so for the inductive step you may assume that $k \geq 2$ and $\ell \geq 2$. Follow the idea of the proof of $10 \to (4,3)$.

8.7 DO: Prove: $4^n / \binom{2n}{n} \sim c\sqrt{n}$. Determine the value of the constant c. — Remember that this quotient is proportional to $\sqrt{n}$; this statement describes a fundamental property of the bell curve and is related to the fact that the standard deviation of the number of Heads among n coin flips is proportional to $\sqrt{n}$.

8.8 HW (6 points): Prove: $k^2 \not\to (k+1, k+1)$. Use the wording of Problem 8.2. Describe the graph G. Do not prove. Your solution should be one line. Longer solutions will not be accepted.

8.9 HW (3+7 points): (a) Define the relation $n \to (k, \ell, m)$. (b) Prove: $17 \to (3, 3, 3)$.

8.10 CHALLENGE (7 points): Prove: $16 \not\to (3, 3, 3)$. Hint: Use the finite field of order 16. Only clearly formulated and elegant proofs will receive (partial) credit. A picture will not be accepted. DO NOT USE web sources.