Graph Theory – CMSC-27500 – Spring 2015
http://people.cs.uchicago.edu/∼laci/15graphs
Homework set #9.   Posted 4-28, 2:40pm
Due Thursday, April 30, typeset in **LaTeX**.

Do not submit homework before its due date; it may get lost by the time we need to grade them. If you must submit early, write the early submissions on separate sheets, separately stapled; state "EARLY SUBMISSION" on the top, and send email to the instructor listing the problems you submitted early and the reason of early submission. **Read the homework instructions on the website.** The instructions that follow here are only an incomplete summary.

Hand in your solutions to problems marked "HW" and "BONUS." Do not hand in problems marked "DO." Warning: the BONUS problems are underrated. PRINT YOUR NAME ON EVERY SHEET you submit. **Use LaTeX to typeset your solutions**. (You may draw diagrams by hand.) Hand in your solutions on paper, do not email. If you hand in solutions to CHALLENGE problems, do so on a **separate sheet**, clearly marked "CHALLENGE," and notify the instructor by email to make sure it won't be overlooked.

Carefully study the policy (stated on the website) on collaboration, internet use, and academic integrity. **State collaborations and sources both in your paper and in email to the instructor.**

Definitions, notation. As before $G = (V, E)$ denotes a graph with n vertices and m edges.

9.1 DO: Review the proof of the following result of Erdős, proved in class: for almost every graph G we have
$\alpha(G) \le 1 + 2\log_2 n$ and $\omega(G) \le 1 + 2\log_2 n$.

9.2 DO: Prove: for almost all graphs,

$$\chi(G) > (\omega(G))^{100}$$

9.3 DO (Zsigmond Nagy's explicit Ramsey graph): We define Nagy's graph as follows. Let $v \ge 3$ be an integer and $n = \binom{v}{3}$. Consider the graph G of which the vertices are the 3-subsets of $[v]$; and two vertices A and B are adjacent exatly if $|A \cap B| = 1$. Prove: (a) $\alpha(G) \le v$ and (b) $\omega(G) \le v$. So Nagy's graph explicitly demonstrates the relation $\binom{v}{3} \not\to (v+1, v+1)$. (c) This implies $n \not\to (cn^{1/3}, cn^{1/3})$ for some positive constant c and all sufficiently large n. Determine the largest c for which this inference is correct.

9.4 DO (a) Study Jensen's inequality from LN and other sources. (b) Prove Jensen's inequality. (c) Derive the inequality of the arithmetic and the geometric mean from Jensen's inequality. (d) Derive the inequality of the arithmetic and the quadratic mean from Jensen's inequality. (e) Derive the inequality of the arithmetic and the quadratic mean from Cauchy–Schwarz.

9.5 DO (Turán's Theorem) Let $T_{n,r}$ denote the complete r-partite graph with n vertices as evenly distributed among the r parts as possible (the size of each pair of parts differs by at most 1). (This is called "Turán's graph," hence the letter T.) Prove: If G is a graph with n vertices and $G \not\supset K_{r+1}$ then $|E(G)| \le |E(T_{n,r})|$. (Review the proof from class.)

9.6 DO: Fix the value of r while $n \to \infty$. Prove:
$|E(T_{n,r})| \sim (n^2/2)(1 - 1/r)$.

9.7 CH (6 points): Let $A_1, \dots, A_m$ be convex subsets of $\mathbb{R}^n$. Assume every $n + 1$ of the A_i have non-empty intersection. Prove that all the A_i have non-empty intersection.

9.8 HW (6 points): A *partition* (A, B) of the set V is a representation of V as $V = A \dot\cup B$ where A, B are non-empty disjoint sets. (The latter condition is indicated by the dot over the $\cup$.) A *cut* (A, B) of a digraph $G = (V, E)$ is a partition (A, B) of V such that there is no edge from A to B. Prove: the digraph G is strongly connected if and only if it has no cut.