Graph Theory CMSC-27500 Final midterm. June 2, 2015
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Name: ____

Show all your work. **Do not use book, notes, or scrap paper.** The use of **electronic devices is strictly prohibited.**

Write your answers in the space provided. You may **continue on the reverse.** The BONUS problems are underrated, solve the non-BONUS problems first.

This test contributes 42% to your course grade.

As usual, $G = (V, E)$ is a graph or a digraph, n denotes the number of vertices, m the number of edges. If G is a graph then $\alpha(G)$ denotes the independence number (maximum number of independent vertices), $\omega(G)$ the clique number (maximum number of pairwise adjacent vertices), $\chi(G)$ the chromatic number, $\nu(G)$ the matching number (maximum number of disjoint edges), and $\tau(G)$ the covering number (minimum number of vertices that hit every edge). We usually omit G from the notation and just write ν for $\nu(G)$, α for $\alpha(G)$, etc.

When referring to a *bipartite graph* $G = (V, E)$, we shall assume a bipartition $V = R \dot\cup B$ (R: "red vertices," B: "blue vertices"; the dot indicates that these sets are disjoint). We say that R can be (fully) matched if $\nu = |R|$. For a set $S \subseteq R$ let $N(S)$ denote the set of neighbors of S, i. e., the set of vertices $v \in B$ such that $(\exists w \in S)(v \sim w)$. Recall that the Marriage Theorem says: R can be fully matched if and only if for every subset $S \subseteq R$ we have $|S| \leq |N(S)|$.

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1. (30 points) State a "good characterization" theorem in as many of the following topics as you can: (a) perfect matchings in general (not necessarily bipartite) graphs; (b) DAGs; (c) graph coloring (but not "2-colorability"); (d) Hamiltonicity; (e) Ramsey Theory; (f) planarity; (g) internally vertex-disjoint paths. — State at most one result from each area. You gain points by precisely stating relevant results, including the name of the person whose result it is; you lose points by stating results that are not good characterizations.

2. (10 points) True or false: "Let $s \neq t$ be vertices of the graph G. If each pair of $s - t$ paths shares an edge then all $s - t$ paths share an edge." Clearly state your answer and prove it. You may use a result proved in class; state the result.

3. (22 points) For every $k \geq 3$, construct a graph that does not contain K_k, has chromatic number $\geq k + 1$, and has $O(k)$ vertices. (Hint: first solve for $k = 3$.)

4. (25+13 points) Let $G = (V, E)$ be a bipartite graph with bipartition $V = R \dot{\cup} B$. Assume $|R| = |B| = t \geq 1$ and let $R = \{u_1, \ldots, u_t\}$ and $B = \{v_1, \ldots, v_t\}$. Define the $t \times t$ *incidence matrix* $M = (m_{ij})$ by setting $m_{ij} = 1$ if $\{u_i, v_j\} \in E$ and $m_{ij} = 0$ otherwise. (a) Prove: If $\det(M) \neq 0$ then M has a perfect matching. (b) Prove that the converse is false; give a small counterexample.

5. (25+12 points) For a graph G and a positive integer t let $P_G(t)$ denote the number of functions $f : V \to [t]$ that are legal colorings of G. (a) Prove that $P_G(t)$ is a polynomial (called the "chromatic polynomial" of G). (b) Compute $P_T(t)$ when T is a tree. Prove your answer.

6. (6+6+5+12B points) (a) Let $\mathrm{DAG}(G)$ denote the number of acyclic orientations of G. Prove: $\mathrm{DAG}(G) \le 2^m$. (b) Characterize those graphs for which $\mathrm{DAG}(G) = 2^m$. (c) Compute $\mathrm{DAG}(C_n)$. (d) (BONUS) Prove: $P_G(-1) = (-1)^n \,\mathrm{DAG}(G)$ (Stanley's Theorem).

7. (BONUS: 6B+12B points) (a) Prove: a graph G is k-colorable if and only if it has an acyclic orientation H such that H has no paths of length k. (b) Prove: If G has an orientation L such that L has no paths of length k then G is k-colorable. (Note that L may have cycles.)

8. (16+8 points) Consider the Erdős–Rényi model $\mathbf{G}_{n,p}$ of random graphs: on a fixed set of n vertices, we decide adjacency by flipping a biased coin that gives "adjacent" with probability p independently for each pair of vertices. (a) Determine the expected number of 5-cycles. Your answer should be a very simple formula. Do not prove. (b) Prove: For $t \geq 3$, the expected number of t-cycles is less than $(np)^t$.

9. (25 points) Let G be a random graph from the $\mathbf{G}_{n,p}$ distribution where $0 < p \leq 1/2$. Prove: almost surely

$$\alpha(G) \leq 1 + \frac{2 \ln n}{p}.$$

10. (BONUS: 6 points) Use the previous two problems to prove Erdős's theorem: For all g and k there exists a graph of girth $\geq g$ and chromatic number $\geq k$.

11. (8+14 points) Prove: for every $\epsilon > 0$, almost all graphs satisfy (a) $\omega < n^\epsilon$ and (b) $\chi > n^{1-\epsilon}$. (You may use other problems from this test.)

12. (4+30+7B points) A graph is *tough* if for every non-empty subset $S \subseteq V$, the graph $G - S$ has at most $|S|$ connected components. (a) Prove: If G is Hamiltonian (has a Hamilton cycle) then G is tough. (b) Prove: if G is a regular graph of degree $d \geq 1$ and G is d-edge-connected then G is tough. (c) (BONUS) Prove: there exists a tough graph that is not Hamiltonian. — Hint: the Petersen graph. Draw it with a 3-fold rotational symmetry to cut down the number of cases.

13. (25 points) Prove: every graph G has a 3-colorable subgraph with $\geq 2m/3$ edges. Use the probabilistic method. State the size of your sample space. Give a clear definition of your random variables.

14. (27 or 16 points) Prove ONE of the following. (a) $9 \to (4,3)$. (b) $10 \to (4,3)$. — Use without proof that $6 \to (3,3)$.

15. (28 points) Prove that every triangle-free planar graph is 4-colorable. Use the fact that every triangle-free planar graph with $n \geq 3$ vertices has $m \leq 2n - 4$ edges. Make your solution algorithmic; make it clear in what order you assign colors to the vertices.

16. (5+25 points) (a) State Tutte's necessary and sufficient condition for the existence of a perfect matching. Do not prove. (b) Deduce the Marriage Theorem (see Preamble) from Tutte's Theorem under the assumption $|R| = |B|$ (the special case when the Marriage Theorem guarantees a perfect matching).

17. (15+8+6B points) Let G be a graph. Recall that $\nu \le \tau \le 2\nu$. (a) Find a non-bipartite graph satisfying $\tau = \nu = 2$. (b) For every $k \ge 1$ find a **connected** graph G such that $\nu = k$ and $\tau = 2k$. Just state the examples, do not prove. (c) (BONUS) Find a triangle-free non-bipartite graph with $\tau = \nu = 3$. [Typo corrected after test; originally the request was $\tau = \nu = 2$, which is impossible.]

18. (28 or 18 points) Prove ONE of the following: (a) If the graph G does not contain $K_{3,3}$ as a subgraph then $m = O(n^{5/3})$. (b) If G does not contain $K_{2,3}$ then $m = O(n^{3/2})$. (Note: these are special cases of a more general theorem by Kővári, Turán, and Sós.) [Two bad typos corrected after the test; originally the exponents stated were 2/3 and 1/2]

19. (BONUS: 8 points) A digraph is called *Eulerian* if for each vertex v we have $\deg^+(v) = \deg^-(v)$. Prove: If an Eulerian digraph is weakly connected then it is strongly connected. ("Weakly connected" means connected as an undirected graph (ignoring the orientation of the edges).)

20. (BONUS: 12 points) Recall *Turán's graph* $T(n,k)$: it has n vertices divided up into k parts $V_1, \dots, V_k$ as evenly as possible; two vertices are adjacent exactly if they don't belong to the same part. Let $m(n,k)$ denote the number of edges of $T(n,k)$. — Prove Turán's Theorem: If $G \not\supseteq K_{k+1}$ then $m \le m(n,k)$.

21. (BONUS: 15 points) Prove that Nagy's explicit Ramsey graph demonstrates $\binom{v}{3} \not\to (v+1, v+1)$. Recall the construction: the $n = \binom{v}{3}$ vertices are labeled by 3-subsets of $[v]$; the vertices corresponding to subsets A and B are adjacent if $|A \cap B| = 1$.

!!! FOR GRADERS ONLY — DO NOT WRITE ON THIS PAGE !!!

1.	/30	12a	/4
2.	/10	12b	/30
3.	/22	12c	/7B
4a	/25	13	/25
4b	/13	14	/27
5a	/25	15	/28
5b	/12	16a	/5
6a	/6	16b	/25
6b	/6	17a	/15
6c	/5	17b	/8
7a	/6B	17c	/6B
7b	/12B	18	/28
8a	/16	19	/8B
8b	/8	20	/12B
9.	/25	21	/15B
10	/6B	• .	
11a	/8	• .	
11b	/14		