Graph Theory CMSC-27500 First Quiz. April 7, 2015

Instructor: László Babai

Name: ____

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may **continue on the reverse.**

In your proofs, you may use facts proved in class or stated in HW exercises. If you want to use a fact stated but not proved in class or in written homework, you need to prove that fact, unless specified otherwise in the problem. (This includes all DO exercises not proved in class.)

The Bonus problem is underrated, solve the non-Bonus problems first.

This quiz contributes 5% to your course grade.

1. (3+9 points) **(a)** Define what is a *relation* on a set A. **(b)** Let $|A| = n$. Count the relations on A. Your answer should be a very simple formula; do not prove.

2. (4+19 points) **(a)** For every $n \geq 3$, name a connected graph G with n vertices that has a vertex v such that $G \setminus v$ is disconnected. Draw the graph and mark v. **(b)** Prove that every connected graph G with $n \geq 2$ vertices has a vertex v such that the graph $G \setminus v$ is connected. (The graph $G \setminus v$ is obtained from G by removing vertex v and all edges incident with v.) – You may use any of the exercises stated in Homework set #2 (attached) without proof. With the right choice of exercises used, your proof should be no more than three lines.

3. (15 points) Let $\ell \geq 5$. Determine the independence number $\alpha(G)$ of the $5 \times \ell$ toroidal grid $G = C_5 \,\square\, C_\ell$. Prove your answer. (The *independence number* $\alpha(G)$ is the size of the largest independent set of vertices in G.) (Hint: try $\ell = 5$ first.)

4. (BONUS: 10 points) Let G be a regular graph of degree $k \geq 1$ (i. e., every vertex has degree k). Prove: $\alpha(G) \leq n/2$.

APPENDIX

Homework set #2. Due Tuesday, April 7, 2015 (except where otherwise stated)

In all problems, unless otherwise stated, we have a graph $G = (V, E)$ with n vertices and m edges. The graph $G \setminus v$ is defined by deleting the vertex v from G along with the edges incident with v. In other words, $G \setminus v$ is the induced subgraph of G on the vertex set $V \setminus \{v\}$. – A *tree* is a connected graph with no cycles. – The *girth* of G is the length of its shortest cycle. (If there is no cycle in G then its girth is ∞.) For additional definitions and notation, see also the first HW set.

2.1 DO **(Equivalence classes)** Prove: Every equivalence relation arises from a partition. — More formally, this means the following. Let $\Pi = (B_1, \dots, B_k)$ be a partition of the set A, i. e., $A = B_1 \cup \cdots \cup B_k$ where $B_i \neq \emptyset$ and $(\forall i \neq j)(B_i \cap B_j = \emptyset)$. The B_i are the *blocks* of this partition. Then Π defines an equivalence relation on A: for $x, y \in A$ we set $x \sim_\Pi y$ if x, y belong to the same block, i. e., $(\exists i)(x, y \in B_i)$. (Verify that $\sim_\Pi$ is an equivalence relation.) What you need to prove is that if R is an equivalence relation on A then there is a partition Π of A such that $R = \sim_\Pi$. — The blocks of this partition are called the *equivalence classes* of R.

2.2 DO: (a) Prove: congruence modulo m is an equivalence relation on the set of integers. (b) The equivalence classes defined by this relation are called the mod m *residue classes.* Prove that for $m \neq 0$ the number of mod m residue classes is $|m|$.

2.3 DO: (a) Prove: if there is an odd closed walk in G then there is an odd cycle in G. (b) Use this to prove that a graph without odd cycles is bipartite.

2.4 DO: Prove: Every tree with $n \geq 2$ vertices has a vertex of degree 1.

2.5 DO: Prove: If G is a connected graph and v is a vertex of degree 1 then the graph $G \setminus v$ is also connected.

2.6 DO: Use Problems 2.5 and 2.6 to prove that for a tree, $m = n - 1$.

2.7 DO: Prove: a graph is a tree if and only if there is a unique path between each pair of vertices.

2.8 DO: We say that the graph T is a *spanning tree* of the graph G if (a) T is a subgraph of G; (b) $V(T) = V(G)$; and (c) T is a tree. — Prove that G has a spanning tree if and only if G is connected.

2.9 DO: Prove: If G is connected then $m \geq n - 1$.

2.10 DO: Prove: G is a tree if and only if (a) G is connected, and (b) $m = n - 1$.

2.11 DO: Prove: G is a tree if and only if (a) G has no cycles and (b) $m = n - 1$.

2.12 HW (8 points): Draw all nonisomorphic trees with 7 vertices. Clearly state how many you found. (Lose 3 points for each mistake – for missing an isomorphism type and for drawing two isomorphic graphs.)

2.13 HW (6 points): Let T be a tree with n vertices. Determine $P_T(x)$ (the chromatic polynomial of T). Your answer should be a simple closed-form expression (no summation or product symbls, no dot-dot-dots). (Hint: first solve this problem for the case when T is the path of length $n - 1$.)

2.14 DO (due Thursday, April 9): Prove: if G is connected then every pair of longest paths in G intersect (share a vertex).

2.15 BONUS (6 points) Prove: In a tree, all longest paths share a vertex.

2.16 HW (12 points, due Thursday, April 9): Prove: if G is triangle-free then $\chi(G) = O(\sqrt{n})$, i. e., prove that there exists a constant C such that for all sufficiently large n we have $\chi(G) \le C\sqrt{n}$. Explicitly state the smallest constant C for which you can prove this. (Review the big-Oh notation from the instructor's online lecture notes in DM and/or from Rosen.)

2.17 HW (2+2+6 points): Recall that a *matching* in a graph is a set of pairwise disjoint edges. A matching $M \subseteq E$ is *maximal* if no edge can be added to M, i. e., if M is not a proper subset of any matching. A *maximum* matching is a matching of maximum size (largest matching); the *matching number* $\nu(G)$ denotes size of a maximum matching (maximum number of edges in a matching). (a) Determine $\nu(P_n)$ (recall: P_n is the path of length $n-1$) (b) Determine $\nu(C_n)$ (c) Let M be a *maximal* matching in G. Prove: $|M| \ge \nu(G)/2$.

2.18 DO: What is the size of the smallest maximal matching in C_{3k} ? Prove your answer.

2.19 DO: Determine the matching number of (a) the $k \times \ell$ grid (b) the $k \times \ell$ toroidal grid.

2.20 DO: Let G be a regular graph of degree k with girth ≥ 5. (a) Prove: $n \ge k^2+1$. (b) Prove: $n = k^2 + 1$ is possible for $k = 1, 2, 3$ (find such graphs). Note: $n = k^2 + 1$ is also possible for $k = 7$ ("Hoffman – Singleton graph"). The Hoffman–Singleton Theorem (1960) says that the only other conceivable value of k for which $n = k^2 + 1$ could happen is $k = 57$. However, no such graph of degree 57 has been found.

2.21 HW (8 points, due Thursday, April 9): Assume that every vertex of G has degree ≥ 3. Prove that the girth of G is $O(\log n)$.

2.22 BONUS (8 points, due Thursday, April 9) Prove: If G is C_4-free (has no 4-cycles) then $m = O(n^{3/2})$.

2.23 CHALLENGE (8 points, no deadline) Prove that the $O(n^{3/2})$ bound in the preceding exercise is tight, i. e., show that there exists a positive constant c such that for infinitely many values of n (in fact, for all sufficiently large n) there exists a C_4-free graph satisfying $m \ge cn^{3/2}$.