Graph Theory CMSC-27500 Second Quiz. April 21, 2015
Instructor: László Babai

Name: ______________________________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may **continue on the reverse.**

In your proofs, you may use facts proved in class or stated in HW exercises. If you want to use a fact stated but not proved in class or in written homework, you need to prove that fact, unless specified otherwise in the problem. (This includes all DO exercises not proved in class.)

The Bonus problems are underrated, solve the non-Bonus problems first.

This quiz contributes 7% to your course grade.

1. (6 points) Determine the maximum possible number of edges of a DAG (directed acyclic graph) with n vertices. Prove your answer.

2. (12 points) For every $k \geq 1$ find a **connected** graph G such that $\nu(G) = k$ and $\tau(G) = 2k$. (Notation: $\nu(G)$ denotes the matching number (maximum number of disjoint edges); $\tau(G)$ denotes the covering number (minimum number of vertices that hit every edge).) Just describe the examples, do not prove.

3. (26 points) Prove: "half of every graph is bipartite." More precisely, let $G = (V, E)$ be a graph. Prove that there exists a set $F \subseteq E$ such that $|F| \geq |E|/2$ and the graph $H = (V, F)$ is bipartite. Use the probabilistic method. State the size of the sample space of the experiment you use. Give a clear definition of the variables you introduce.

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4. (26 points) Prove, for every k, that almost every tournament is k-paradoxical. Define what "almost every" means. – Explanation: Let $T = (V, E)$ be a tournament with $n = |V|$ vertices. We say that a vertex $v \in V$ *dominates* a subset $A \subseteq V$ if $v \notin A$ and v beats all vertices in A, i. e., $(\forall w \in A)((v \to w) \in E)$. We say that T is *k-paradoxical* if $n > k$ and every subset $A \subseteq V$ of size $|A| = k$ is dominated by some vertex.

5. (BONUS: 10 + 3 + 1 points) Let $\mathrm{DAG}(G)$ denote the number of acyclic orientations of the graph $G = (V, E)$. (a) Prove the deletion/contraction recurrence that states that for every edge $e \in E$,

$$\mathrm{DAG}(G) = \mathrm{DAG}(G - e) + \mathrm{DAG}(G/e)$$

where $G - e$ denote the graph G with the edge e deleted (but no vertices deleted) and G/e denotes the graph G with the edge e contracted (so the number of vertices is reduced by 1). (b) State the theorem to which part (a) is a lemma. Explain the notation used in the theorem. Do not prove. (c) Whose theorem?

6. (BONUS: 9 points) Prove: for every even n, the complete graph K_n has a 1-factorization. — Explanation: you need to prove that the set of egdes of K_n is the union of $n - 1$ perfect matchings. — To gain any credit at all, your construction must be clear and convincing.