Graph Theory CMSC-27500 Third Quiz. May 5, 2015
Instructor: László Babai

Name: ____

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may **continue on the reverse.** The Bonus problems are underrated, solve the non-Bonus problems first.

This quiz contributes 7% to your course grade.

You may use the following results proved in class. As usual, n denotes the number of vertices, m the number of edges, $\alpha(G)$ the independence number (maximum number of independent vertices), $\omega(G)$ the clique number (maximum number of pairwise adjacent vertices), and $\chi(G)$ the chromatic number of the graph G.

Theorem 1 (Erdős). For almost all graphs G we have $\alpha(G) \le 1 + 2\log_2 n$.

Theorem 2 (Kőváry–Turán–Sós). If the graph G has no 4-cycles then $m = O(n^{3/2})$.

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1. (10 points) Let $r \ge 2$. True or false: If the graph G does not contain K_{r+1} then G is r-colorable. — Clearly state and prove your answer. Your proof should be explicit; no results from class should be used. Note that this is a separate question for every r; solving it for a particular value of r earns you partial credit. (Hint: try $r = 2, 3$ first.)

2. (8 points) What is the probability that a random tournament on a given set of n vertices is a DAG? Your answer should be a very simple formula. Do not prove.

3. (5 points) For every $k \ge 1$ find a **connected** graph G such that $\nu(G) = k$ and $\tau(G) = 2k$. (Notation: $\nu(G)$ denotes the matching number (maximum number of disjoint edges); $\tau(G)$ denotes the covering number (minimum number of vertices that hit every edge).) Just state the examples, do not prove.

4. (20 points) Prove: If the graph G has no 4-cycles then $\chi(G) = O(\sqrt{n})$.

5. (21 points) Prove: for almost all graphs G we have $\chi(G) > (\omega(G))^{100}$. (See notation in preamble.)

6. (6 points) Prove: "half of every graph is bipartite." More precisely, let $G = (V, E)$ be a graph. Prove that there exists a set $F \subseteq E$ such that $|F| \geq |E|/2$ and the graph $H = (V, F)$ is bipartite. Use the probabilistic method. State the size of the sample space of the experiment you use.

7. (BONUS: 6 points) Let G be a loop-free digraph and let H be the directed line-graph of G. Prove: $\chi(G) \le 2^{\chi(H)}$.

8. (BONUS: 8 points) Prove: If the graph G has no 5-cycles then $\chi(G) = O(\sqrt{n})$.

9. (BONUS: 8 points) Prove: If every vertex of G has degree $\ge n/2$ then G is Hamiltonian.

10. (BONUS: 8 points) Fix an integer $r \ge 5$. Construct a graph G such that (a) $\chi(G) = 6$ (b) for every edge e we have $\chi(G - e) = 5$ (c) every vertex of G has degree $\ge r$. Hint. Assume r is odd. Make G have $2(r-2)$ vertices and be regular of degree r.