Graph Theory CMSC-27500 Fourth Quiz. May 19, 2015

Instructor: László Babai

Name: ____

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may **continue on the reverse.** The Bonus problems are underrated, solve the non-Bonus problems first.

This quiz contributes 7% to your course grade.

As usual, $G = (V, E)$ is a graph or a digraph, n denotes the number of vertices, m the number of edges. If G is a graph then $\nu(G)$ denotes its matching number (maximum number of disjoint edges), and $\tau(G)$ the covering number (minimum number of vertices that hit every edge).

You may use the following results proved in class or assigned as an exercise: If G is a planar graph and $n \geq 3$ then $m \leq 3n - 6$. If, in addition, G is triangle-free, then $m \leq 2n - 4$.

---

1. (4 points)[1 minute] What is the probability that a random tournament on a given set of n vertices is a DAG? Your answer should be a very simple formula. Do not prove.

2. (4 points) [1 minute] For every $k \geq 1$ find a **connected** graph G such that $\nu(G) = k$ and $\tau(G) = 2k$. Just state the examples, do not prove.

3. (7 points) [2 minutes] Prove: there exists a graph that is not planar and has girth 100. (Recall: the *girth* is the length of the shortest cycle.) Describe the example, do not prove.

4. (8 points) [3 minutes] True or false: "Let $s \neq t$ be vertices of the graph G. If each pair of $s - t$ paths shares an edge then all $s - t$ paths share an edge." Clearly state your answer and prove it. You may use a result proved in class; state the result.

5. (8 points) Find a graph G with special vertices $s \neq t$ such that (a) there exist 5 internally vertex-disjoint $s - t$ paths; but (b) there is an $s - t$ path P such that no $s - t$ path is internally disjoint from P (so P alone is a maximal set of internally vertex-disjoint $s - t$ paths). Draw a clear picture, highlighting P. No explanation is needed.

6. (13 points) Let (G, s, t, c) be a network where $G = (V, E)$ is a digraph, $s \neq t$ are two vertices, and $c : E \to \mathbb{R}$ is the capacity function. Prove that G has a sub-DAG H containing s and t such that the value of the maximum $s \to t$ flow is the same for G and H.

7. (8+8 points) Prove: (a) Every planar graph has a vertex of degree ≤ 5. (b) Prove: every planar graph is 6-colorable. You may use results proved in class; state the results used.

8. (10 points) Prove: There exists a connected regular graph G of degree 100 such that the connectivity of G is $\kappa(G) = 1$.

9. (BONUS: 9 points) Let $k \geq 4$. Prove that the $k \times k$ toroidal grid is not planar. You may use any result stated in class or in homework sheets. State what you use. The solution should be only a couple of lines.

10. (BONUS: 8 points) A *bridge* in a connected graph G is an edge e such that $G-e$ is disconnected. (For instance, every edge of a tree is a bridge.) Prove: if G is a regular graph of degree 100 (i. e., every vertex has degree 100) then then G cannot have a bridge.

11. (BONUS: 5+5 points) Prove: (a) Almost all graphs are not planar. (b) For all sufficiently large n, the probability that a random graph on n vertices is planar is less than $2^{-0.49n}$.

12. (BONUS: 5 points) Prove: If both the graph G and its complement $\overline{G}$ are planar then $n \leq 10$.

13. (BONUS: 5 points) Let $G = (V, E)$ be a k-connected graph and S, T two disjoint subsets of V of size k each. Prove: there exist k disjoint paths each connecting a vertex of S to a vertex of T. (Note: these paths must be disjoint, not just internally disjoint.) You may use results proved in class; state the results used.

14. (BONUS: 2+9 points) (a) Find a planar graph that is regular of degree 3 and has girth 5. (b) Prove: If every vertex of the graph G has degree ≥ 3 and G has girth ≥ 6 then G is not planar. (The girth is the length of the shortest cycle.)