Lecture Notes of Honors Combinatorics

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WARNING: With the exception of the class of May 26, these notes have NOT been proofread by the instructor. They contain many mistakes. Read these notes critically; use them at your own risk.

1 Tuesday, March 29, 2016

Things expected to be known:

- modular arithmetic (congruences modulo m)
- asymptotic notation $a_n \sim b_n \iff a_n = O(b_n)$
- finite probability spaces: expected value and independence of random variables
- basic linear algebra (rank, determinant, eigenvalues)

[See online lecture notes.]

Graph: $G = (V, E)$, where V is the set of **vertices** (singular: vertex) and E is the set of **edges** (unordered pairs of vertices)

(e.g., $V = \{1, \dots, 5\}, E = \{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}; |V| = 5, |E| = 6$)

Bipartite graph: $V = V_1 \cup V_2$

Hypergraph: $\mathcal{H} = (V, \mathcal{E}), \mathcal{E} = \{A_1, \dots, A_m\}, A_i \subseteq V$, [$A_i = A_j$ is permitted.]

Hypergraph $\iff$ bipartite graph; edges: if $v_i \in A_j$

If $x \in V$, then $\deg(x) := \#\{i \mid x \in A_i\}$

Handshake Theorem: $\displaystyle\sum_{x \in V} \deg(x) = \sum_{i=1}^{m} |A_i|$

Proof: Count pairs; ["Actuary Principle"]
$\#\{(x, i) | x \in A_i\}$

$=: \sum_x \deg(x)$ (counting by x)
$=: \sum_i |A_i|$ (counting by i)

Incidence matrix: $n \times m$ matrix, $M_{ij} = \begin{cases} 1 \text{ if } x_i \in A_j \\ 0 \text{ otherwise} \end{cases}$

Regular Hypergraph: if every vertex has same degree $r \iff r$-regularity

Uniform Hypergraph: every edge has same number of sites $k \iff k$-uniformity

Graph (alt.): 2-uniform hypergraph without multiple edges

Notation: $n = \#\text{vertices}$, $m = \#\text{edges}$

r-uniform, k-regular hypergraph $\implies nk = mr$; for graphs, $n \cdot k = 2m$

Hypergraph without multiple edges then $m \leq 2^n$

Intersecting hypergraph: every pair of edges intersects

HW: For intersecting hypergraphs, prove $m \leq 2^{n-1}$ (2-line proof)

k-uniform hypergraph: maximum #edges $= \binom{n}{k}$

k-uniform, intersecting: max $= \binom{n-1}{k-1}$? – Only half of the time!

HW: For every $n \geq 3$ and a lot of values of k: find intersecting k-uniform hypergraphs with $> \binom{n-1}{k-1}$ edges (i.e. lot of values $\iff \to \infty$ as $n \to \infty$)

CH: If $\mathcal{H}$ is regular, of deg ≥ 1, k-uniform, intersecting $\implies k > \sqrt{n}$

Def: A **finite projective plane** is a hypergraph such that

(i) all pairs of points have exactly one line (i.e. hyperedge) through them
(ii) all pairs of lines are intersecting at exactly one point
(iii) (nondegeneracy axiom) $\exists$ four points with no 3 on a line

(e.g., Fano plane, 3-regular, 3-uniform, with 7 points and 7 lines)

[Turn in HW problems. CH problems have no strict deadline, but they are over when discussed in class; email when working on them as a warning, so Babai will avoid discussing in class. Don't turn in DO problems]

DO: A **finite projective plane (alt.)** is:

(a) regular (r-regular)
(b) uniform (k-uniform)
(c) $k = r := n + 1$
(d) $\#\text{points} = \#\text{lines} = n^2 + n + 1$

Projective geometry over $\mathbb{R}$:

e.g., $x := (x_1, x_2, x_3) \sim \lambda(x_1, x_2, x_3) =: \lambda x$;
points are equivalence classes of triples.
Take $\mathbb{R}^3 - \{0\}$. point p has coordinates $(x_1, x_2, x_3) \to$ homogeneous coordinates
Projective line: corresponds to 2-dim subspace of $\mathbb{R}^3$, homogeneous coordinates of a line: a;
$a \cdot x = a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$

$\mathbb{F}$, **Finite Field**: finite #elements with 2 operations $+, \cdot$ satisfying usual axioms (like in $\mathbb{R}$)

$|\mathbb{F}| = q$: A finite field of order q exists $\iff q = p^k$ prime power

e.g., $3 \cdot 3 \equiv 0 \mod 9$

Galois Fields (finite fields), $GF(q) = \mathbb{F}_q$; $\forall q, \exists!$ Galois field of order q

$PG(2, q)$: projective plane over $GF(q)$

#points:
$|\mathbb{F}_q^3| = q^3$ (#triples (x_1, x_2, x_3))
$|\mathbb{F}_q^3 - \{\underline{0}\}| = q^3 - 1$
$|\mathbb{F}_q - \{0\}| = q - 1$ size of equivalence classes
#points = #lines = #equivalence classes = $\dfrac{q^3 - 1}{q - 1} = q^2 + q + 1$

OPEN: For what values of n (i.e. set of these numbers named $\mathcal{P}$) does there exist a projective plane of order n?

We know: If q is a prime power, then $q \in \mathcal{P}$
Also know: infinitely many values of $n \in \mathcal{P}$
No n that is not a prime power is known to belong to $\mathcal{P}$
Smallest n of unknown status: $n = 12$
$6 \notin \mathcal{P} \sim 1990$, tedious
$10 \notin \mathcal{P} \sim$ 199X

$\mathbb{F}_p$, p prime: field; $\mathbb{F}_p[i] = \{a + bi \mid a, b \in \mathbb{F}_p, i^2 + 1 = 0\}$

Likewise, $\mathbb{C} = \mathbb{R}[i]$

HW: For what p is $\mathbb{F}_p[i]$ a field? $\iff$ When are there no zero-divisors: $z_1 z_2 = 0 \implies z_1 = 0$ or $z_2 = 0$

Experiment! Discover simple pattern by looking at primes < 30

2 Thursday, March 31, 2016

Tuesday, April 5: QUIZ

TA: Joseph Tsong

Office hr: Monday 4:30-5:30, Young 208A

$\mathcal{P} :=$ {orders of finite projective planes}. If q is a prime power field then $q \in \mathcal{P}$.

Bruck-Ryser Theorem:

If $n \equiv 1$ or $2 \mod 4$ and $n \in \mathcal{P}$, then $\exists a, b : n = a^2 + b^2$.
Bruck-Ryser gives us only:
in $\mathcal{P}$: $2, 3, 4, 5, 7, 8, 9, 11, 13$
not in $\mathcal{P}$: $6, 14$
don't know: 12
no 10

Latin Square, $n \times n$: [i.e., solved Sudoku puzzle and superpositions thereof]

3	1	2		11	23	32		1	2	3		1	3	2
2	3	1	,	22	31	13	=	2	3	1	&	2	1	3
1	2	3		33	12	21		3	1	2		3	2	1

E.g., Euler: "36 officers' problem" wanted a pair of orthogonal 6×6 Latin squares

HW: If n is odd ≥ 3, then $\exists$ pair of $n \times n$ orthogonal Latin squares.

DO: If $\exists$ pair of $k \times k$ orthogonal Latin squares and a pair of $l \times l$ orthogonal Latin squares, then $\exists$ pair of $kl \times kl$ orthogonal Latin squares.

DO: If q is a prime power ≥ 3, then $\exists$ pair of orthogonal Latin squares

$\therefore$ If $n \geq 3$ and $n \not\equiv 2 \mod 4$, then $\exists$ pair of orthogonal Latin squares.

Theorem: $\nexists$ pair of 6×6 orthogonal Latin squares.

Bose-Shrikhande-Parker: 6 is the only exception.

DO: (a) If $\exists m$ pairwise orthogonal $n \times n$ Latin squares, then $m \leq n - 1$ (b)$\exists n - 1$ pairwise orthogonal $n \times n$ Latin squares, then $\exists$ projective plane of order n

DO: Dual of a projective plane is a projective plane.

"Possibly degenerate projective plane"

(i) every pair of points is connected by a line
(ii) every pair of lines intersects

(iii) there is a triple of points not on a line.

DO*: the only degenerate projective planes are a bunch of points on a line all also having a line through another point off to the side of the main line.

$\mathcal{H}$: k-uniform intersecting hypergraph

If $\mathcal{H}$ is a possibly degenerate projective plane, p: point, l: line

Lemma: $p \,\underline{\ \cdot\ }\, l \implies \deg(p) = |l|$

(all lines touch $n+1$ points: #points $= 1 + n(n+1) = n^2 + n + 1$)

Galois plane over $\mathbb{F}_q$: points $[a, b, c], a, b, c \in \mathbb{F}_q$ (homogeneous coordinates), not all are zero; $(a, b, c) \sim (\lambda a, \lambda b, \lambda c), \lambda \neq 0$. p: homogeneous coordinates for point p, l: homogeneous coordinates for line l

Claim: p_1, p_2, p_3 not on a line, i.e., if $a \cdot p_i = 0$, $i = 1, 2, 3$, then $\underline{a} = \underline{0}$.

DO: finish

$p \multimap l$ if $p \cdot l = 0$ (i.e., $\sum p_i l_i = 0$)

CH: **Fundamental Theorem of Project Geometry**: If $(p_1, \ldots, p_4)$ and $(q_1, \ldots, q_4)$ are quadruples of points in $PG(2, \mathbb{F})$ in general position (no 3 on a line), then $\exists f$: $\{\text{points}\} \to \{\text{points}\}$ **collineation** such that $f(p_i) = q_i$ (collineation: $\exists f^{-1}$, preserves collinearity)

Theorem ((Paul) Erdös - (Chao) Ko - (Richard) Rado): if $k \leq n/2$ then $m \leq \binom{n-1}{k-1}$

Lemma: $\mathcal{H}$ regular, uniform hypergraph $\mathcal{H} = (V, \mathcal{E}), 0 \leq \alpha \leq 1$, $R \subseteq V$ "red vertices." Assume $\forall A \in \mathcal{E}$, $|R \cap A| \leq \alpha k$. Then, $|R| \leq \alpha n$.

DO: False if we omit regularity:

Prove $\exists$ uniform hypergraph without isolated (i.e., deg $= 0$) vertices (i.e., $\cup \mathcal{E} = V$) and $R \subseteq V$ such that
(a) $\forall A \in \mathcal{E}, |R \cap A| \leq k/10$ and
(b) $|R| \geq 9n/10$.

"Lubell's permutation method"

$S = \{\sigma : \text{cyclic permutations of } V\}$
$\binom{V}{k}$ =set of all k-subsets of V
edge: A is an arc on σ; "A and σ are compatible"
$(\binom{n-1}{k-1} / \binom{n}{k} = k/n)$

DO: Lemma: At most k edges of $\mathcal{H}$ are compatible with a given σ (assuming $k \leq n/2$)

3 Tuesday, April 5, 2016

Recall: If $\mathcal{H}$ is simple, intersecting, then $m \leq 2^{n-1}$. (Simple(st)) Proof: Take V as $A \cup \bar{A}$. Since $A, \bar{A}$ are disjoint, only one can be in the set of hyperedges we construct; we are using Pigeonhole Principle.

Erdös-Ko-Rado: If $k \leq n/2$, intersecting simple k-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$, then $m \leq \binom{n-1}{k-1}$.

Lemma 1: $\mathcal{L} = (W, \mathcal{F})$, k-uniform, regular, red verices $R \subseteq W$, and $0 \leq \alpha \leq 1$. If every edge has $\leq \alpha k$ red vertices then $|R| \leq \alpha n$.

Suppose hypergraph is r-regular. Look at $\{(V, E) \mid v \in R\}$; here, $|\{(V, E) \mid v \in R\}| = r|R| \leq m\alpha k$, so $|R| \leq \alpha mk/r = \alpha n$ (recall $mk = rn$).

Lemma 2: n points in cycle and k-arc(s) where $k \leq n/2$. Set $\mathcal{C}$ of k-arcs that pairwise intersect. Prove: $|\mathcal{C}| \leq k$.

Lemmmas 1 & 2 $\implies$ EKR: take a hypergraph with $\binom{n}{k}$ vertices and $(n-1)!$ edges, which correspond to the cyclic permutations σ of the labels on the vertices. Then, define edges $A \subseteq V$, $|A| = k$ and incidence by: (σ, A) incident if A is an arc on σ. Let us call this new hypergraph $\mathcal{L}$. $\mathcal{L}$ is n-uniform (there are n possible arcs) and regular (by symmetry). Red points can be defined here as the edges in $\mathcal{E}$. By Lemma 2, $\leq \alpha = k/n$ proportion of every edge in $\mathcal{L}$ is red. $\therefore |\mathcal{E}| = |R| \leq |W|k/n$. Here, $|W| = \binom{n}{k}$; $\therefore m = |\mathcal{E}| = |R| \leq \binom{n}{k}k/n = \binom{n-1}{k-1}$. QED (This is Lubell's permutation method.)

Polarity in a projective plane $\mathcal{P} = (P, L, I)$; where $P = \{\text{points}\}, L = \{\text{points}\}$, incidence relation $I = \{(p, l) | p \multimap l\}$; is $f : P \to L$, a bijection, such that $\forall p, q \in \mathbb{P}, p \multimap f(q) \iff q \multimap f(p)$.

HW: Prove: every Galois plane has a polarity.

Finite probability spaces, $(\Omega, \mathbb{P})$:

Ω "sample space": nonempty set,
$\mathbb{P}$ = probability distribution over Ω,
$\mathbb{P} : \Omega \to \mathbb{R}$ such that $\forall a \in \Omega\ \mathbb{P}(a) \geq 0$
a is an elementary event, $\mathbb{P}(a)$ is a probability, and $\sum_{a \in \Omega} \mathbb{P}(a) = 1$.

Outcomes of an experiment: e.g., n coin flips HTTTHTHHT: $|\Omega| = 2^n$, poker hand 5 cards out of the standard deck of 52 cards $\binom{52}{5}$.

Event: $A \subseteq \Omega, \mathbb{P}(A) := \sum_{a \in A} \mathbb{P}(a)$

Uniform distribution: $\forall a \in \Omega, \mathbb{P}(a) = 1/n, n = \Omega$.

If uniform, then $\mathbb{P}(A) = |A|/|\Omega| = \#$"good cases"/$\#$"all cases". $\binom{n}{k}/2^n = \mathbb{P}(k$ heads in n fair coin flips).

Random variable over the probability space $\Omega, \mathbb{P}$ is a function $X : \Omega \to \mathbb{R}$

DEF: **Expected (or Mean) Value** of a random variable $E(X) = \sum_{a \in \Omega} X(a)\mathbb{P}(a) = \sum_{y \in \text{Range}(X)} y \cdot \mathbb{P}(x = y)$ weighted average of the values of X.

"$X = y$"$= \{a \in \Omega \mid X(a) = y\}$

DO: $\min X \leq E(X) \leq \max X$.

Indicator variable: takes value 0 or 1, equivalent to events

θ_A: indicator of event A, $\theta_A(a) = \begin{cases} 1 \text{ if } a \in A \\ 0 \text{ if } a \notin A. \end{cases}$

$|\Omega| = n \implies$ there are 2^n indicator variables.

$E(\theta_A) = 1 \cdot \mathbb{P}(\theta_A = 1) + 0 \cdot \mathbb{P}(\theta_a = 0) = \mathbb{P}(A)$, i.e. "$\theta_A = 1$"=A and $\mathbb{P}(\theta_A = 1) = \mathbb{P}(A)$.

DO: Linearity of expectation: If $X_1, \ldots, X_k$ are random variables over $(\Omega, \mathbb{P})$ and $c_1, \ldots, c_k \in \mathbb{R}$, then $E(\sum c_i X_i) = \sum c_i E(X_i)$

DO: Use this to prove: $E(\#\text{heads in } n \text{ coin flips}) = n/2$. Hint: write $X = \sum_{i=1}^n Y_i$, Y_i: indicator of event "i^{th} flip in heads"

Random **permutations** of a set of S of n elements, bijections $\pi : S \to S$. $|\Omega| = n!$.

<u>Notation</u>: $[n] = \{1, \ldots, n\}$.

E.g., $S = [10]$, $\pi(S) = \{7, 5, 4, 6, 1, 10, 8, 9, 2, 3, 11\}$

HW: Let X be the length of the cycle through point 1. $\mathbb{P}(X = k) = 1/n$.

$\mathbb{P}(X = 1) = (n-1)!/n! = 1/n, \mathbb{P}(X = n) = (n-1)!/n! = 1/n$.

HW: Let Y : #edges. Prove: $E(Y) \sim \ln n$

4 Thursday, April 7, 2016

Consider the cardgame Set, with 81 cards each endowed 4 attributes (i.e. card$\in \mathbb{F}_3^4$): ternary color, number (1,2,3), shape (circle, diamond, squiggle), shading (completely shaded, partially shaded, not shaded)

Set: 3 cards such that in each attribute, either each card same or all different, i.e. $SET = \{(\underline{x}, \underline{y}, \underline{z}) \mid$ distinct components, such that $\underline{x} + \underline{y} + \underline{z} = \underline{0}\}$

Note that $\mathbb{F}_3^n := \{(x_1, \dots, x_n) \mid x_i \in \mathbb{F}_3\}$

HW: n-dimensional "SET" is a 3-uniform hypergraph with 3^n vertices. Assuming that it is regular, what is the degree of its vertices?

$\mathcal{H} = (V; A_1, \dots, A_m), W \subseteq V$ **independent** if $\forall i, A_i \not\subseteq W$, $\alpha(\mathcal{H})$ =independence number=size of largest independent set

$\alpha_k := \alpha(n\text{-dim SET game})$

HW: $\alpha_{k+l} \geq \alpha_k \alpha_l$

DO: **Fekete's Lemma**: If $\{a_n\}$ is super multiplicative ($a_n > 0$), i.e. $a_{k+l} \geq a_k a_l$, then $\exists \lim_{n\to\infty} \sqrt[n]{a_n} = \sup_n\{\sqrt[n]{a_n}\}$

Corollary: $\exists \lim \sqrt[n]{\alpha_n} = \sup \sqrt[n]{\alpha_n} =: L$; $2^n \leq \alpha_n \leq 3^n, 2 \leq L \leq 3$

HW: $L > 2$. (You may use information on ordinary SET game on web.)

OPEN: $L > 3$? Best (Meshulam): $\alpha_n < 2 \cdot 3^n/n$. (Proof: character of finite abelian groups.)

χ: **chromatic number**

legal coloring: no edge becomes monochromatic of vertices

$\mathcal{H}$ hypergraph, **optimal coloring**: $\chi(\mathcal{H}) = \min \#$colors in a legal coloring

HW: (a) $\alpha(\mathcal{H})\chi(\mathcal{H}) \geq n$.

(b) use this to prove: $\chi(n\text{-dim SET}) \to \infty$ (use a result stated)

DO: $\chi(\text{FANO}) = 3$

DO MAYBE: $\chi(PG(2,3), PG(2,4))$, where 3 and 4 corresponds to $\mathbb{F}_3$ and $\mathbb{F}_4$, respectively

HW: (a) If $\mathcal{H}$ is k-uniform and $m \leq 2^{k-1}$ ($k \geq 2$), then $\chi(\mathcal{H}) \leq 2$. (Hint: union bound)

(b) If $\mathcal{P}$ is a projective plane of order $n \geq 5$ then $\chi(\mathcal{P}) = 2$.

Finite probability spaces: $A_1, \dots, A_t \subseteq \Omega$ events

DO: Union bound $\mathbb{P}(\cup_{i=1}^t A_i) \leq \sum_{i=1}^k \mathbb{P}(A_i)$ Proof: induction on t

Events A, B are **independent**: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

A, B, C are independent: pairwise independence and $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$

DO: Find small probability space and 3 events satisfying $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$ but not pairwise independence

DO: A and A are independent $\iff$?

$A_1, \ldots, A_t$ are independent if for $I \subseteq [t] = \{1, \ldots, t\}$, $\mathbb{P}(\cap_{i \in I} A_i) = \prod_{i \in I} \mathbb{P}(A_i)$

2^t conditions $I = \emptyset : \cap_{i \in \emptyset} A_i = \Omega, \prod_{i \in \emptyset} = 1$

$|I| = 1$ true is actually $2^t - t - 1$ conditions

DO: If $A_1, \ldots, A_t$ are independent, then $A_1, \ldots, A_{t-1}$, $\bar{A}_t = \Omega - A_t$ are independent

$\therefore \{A^1 = A, A^0 = \bar{A}\}$, then $A_1^{\xi_1}, \ldots, A_t^{\xi_t}$ are independent for all $\epsilon_i \in \{0, 1\}$

A, B, C independent $\implies A, B \cup C$ independent

DO: Generalize the above to all Boolean combinations $\cup, \cap, -$ with a finite set of events

$X_1, X_2, \ldots, X_t$ random variables over $(\Omega, \mathbb{P})$ are **independent** if $\forall x_1, \ldots, x_t \in \mathbb{R}, \mathbb{P}(X_1 = x_1, \ldots, X_t = x_t) = \prod_{i=1}^t \mathbb{P}(X_i = x_i)$

DO: If $X_1, \ldots, X_t$ are independent, then all their subsets are independent.

DO: events $A_1, \ldots, A_t$ are independent $\iff$ indicator variables $\theta_{A_1}, \ldots, \theta_{A_t}$ are independent

Markov's Inequality: Suppose X is a positive random variable $X \geq 0$, $a > 0$. Then $\mathbb{P}(X \geq a) \leq E(X)/a$

DO: Prove Markov's Inequality in one line

Variance $\mathrm{Var}(X) = E((X - E(X))^2)$. Write $m = E(X)$ for now. Then, we immediately have $\mathrm{Var}(X) = E(X^2) - 2mE(X) + m^2 = E(X^2) - m^2 = E(X^2) - E(X)^2$

Corollary (**Cauchy-Schwartz inequality**): $E(X^2) \geq E(X)^2$

DO: Compare with other forms of the Cauchy-Schwartz inequality

Covariance $\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)$.

$\mathrm{Var}(X) = \mathrm{Cov}(X, X)$

DO: If X, Y are independent, then $E(XY) = E(X)E(Y)$. If $X_1, \ldots, X_t$ are independent, then $E(\prod X_i) = \prod E(X_i)$.

If X, Y independent, then $\mathrm{Cov}(X, Y) = 0$ (i.e., X, Y **uncorrelated**)

HW: Show X, Y independent $\not\Longleftarrow$ $\mathrm{Cov}(X, Y) = 0$ (Make Ω small)

$Y = X_1 + \ldots + X_t$

$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = E(\sum_i \sum_j X_i X_j) - \sum_i \sum_j E(X_i)E(X_j)$

$\quad = \sum_i \sum_j (E(X_i X_j) - E(X_i)E(X_j)) = \sum_i \sum_j \mathrm{Cov}(X_i, X_j)$

$= \sum_i \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j) = \sum_i \mathrm{Var}(X_i) + 2\sum_{i<j} \mathrm{Cov}(X_i, X_j)$

Random graphs

Erdös-Renyi Model with $p = 1/2$: fix V, $|V| = n$. $E(\#\text{edges}) = \binom{n}{2}/2$, $T_n = E(\#\text{triangles})$ (DO: use linearity of expectation)

DO: Find exact formula and find asymptotic value of variance $\mathrm{Var}(T_n) \sim a \cdot n^b$, find a, b

5 Tuesday, April 12, 2016

SUBSTITUTE: Prof. Alexander Razborov

Binomial Theorem: $(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$; here, we will more simply write this without y, $(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$.

Then, we have $\sum_{j=0} \binom{n}{j} = 2^n, \sum_{j=0}(-1)^j \binom{n}{j} = 0$. More cleverly we could also try $x = \sqrt{3}i = \sqrt{-3}$ to find $(1/2 - i\sqrt{3}/2)^3 = 1$, or equivalently $(1 - i\sqrt{3})^{3m} = \pm 2^{-3m}$.

Now, consider $f'(x) : n(1 + x)^{n-1} = \sum_{j=0}^{n} j\binom{n}{j} x^{j-1}$. For example, we find $x = 1 \implies \sum_{j=0}^{n} j\binom{n}{j} = n2^{n-1}$.

Also, we can iterate the formula to obtain $(1+x)^{2n} = \sum_{k=0}^{2n} x^k \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-j}$, from which we can find $\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}\binom{n}{n-j}$.

(Formal) **Power series**: $(a_0, a_1, a_2, \ldots, a_n, \ldots) \to$ Generating function $G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$

For example, $|a_n| \leq ck^n$ where c, k are arbitrary constants. In the case $(c, k) = (-1/k, -1/k)$, we obtain Taylor series

We can do some operations on generating functions:

multiply by constants: $\alpha G_1(x)$
add such functions: $(G_1(x) + G_2(x))$
multiply such functions:
$(a_0 + a_1 x + \ldots)(b_0 + b_1 x + \ldots) = (c_0 + c_1 x + \ldots)$,
where $c_k = a_0 b_k + a_1 b_{k-1} + \ldots + a_k b_0$ (a **Recurrent relation**).
even divide them given $G_2(0) \neq 0, a_0 \neq 0$: $\frac{G_1(x)}{G_2(x)} = \frac{c_0 + c_1 x + c_2 x^2 + \ldots}{a_0 + a_1 x + a_2 x^2 + \ldots} = (b_0 + b_1 x + b_2 x^2 + \ldots)$

Recall the examples:

$G(x) = 1 + x/1! + x^2/2! + x^3/3! + \ldots + x^n/n! + \ldots = e^x$ (here $G'(x) = G(x)$)
$G(x) = x - x^3/3! + x^5/5! \ldots = \sin x$

We have the fact $(1+x)^r = \sum_{j=0}^{\infty} \binom{r}{j} x^j$ and one from analysis: for arbitrary r we can take $(x^r)' = rx^{r-1}$. Note for arbitrary r, as long as j is an integer, we have $\binom{r}{j} = \frac{r(r-1)\cdots(r-j+1)}{j!}$. Using these facts we can find that

$(1+x)^{-n} = \sum_{j=0}^{\infty} \binom{-n}{j} x^j = \sum_j \frac{(-n)(-n-1)\cdots(-n-j+1)}{j!} x^j$

$\quad = \sum_j (-1)^j \binom{n+j-1}{j} x^j = \sum (-x)^{-j} \binom{n+j-1}{n-1}$

Replacing $i = n+j-1$, we can rearrange to obtain $x^{n-1}(1-x)^{-n} = \sum_i x^i \binom{i}{n-1}$, an useful formula which we can apply to find:

$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots$
$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \ldots + nx^n + \ldots$
$\frac{x^2}{(1-x)^3} = \frac{2\cdot 1}{2} x^2 + \frac{3\cdot 2}{2} x^3 + \frac{4\cdot 3}{2} x^4 + \ldots \frac{(n+2)(n+1)}{2} x^{n+2} + \ldots$

Here, we have applications to recurrence relations, Fibonacci numbers($F_0 = 0, F_1 = 1, F_n + F_{n-1}$), and rational functions.

Rational functions: $p(x)/q(x)$, where p, q are polynomial, $\deg p < \deg q$, and $q(x) = 1 - u_1 x - u_2 x^2 - \ldots - u_d x^d$

$G(x)q(x) = p(x)$; we can obtain $a_k = u_1 u_{k-1} + u_d u_{k-d}$

Overall, we have determined "Rational functions≡recurrence relations"

$q(x) = a(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_d)$, $\lambda_1, \ldots, \lambda_d$ are pairwise distinct. Take $\frac{p(\lambda)}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_d)} = \frac{\alpha_1}{x-\lambda_1} + \ldots + \frac{\alpha_d}{x-\lambda_d}$ to express generating functions

So, we can approach the Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ with a formula. Note $\frac{x}{1-x-x^2} = \frac{c_0}{x-\lambda_1} + \frac{c_1}{x-\lambda_2}$. We can obtain $F_n = (\lambda_1^n - \lambda_2^n)/\sqrt{5}$, where $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_2 = (1-\sqrt{5})/2$

Now, let us consider the Drunkard's/Random Walk: we want to find a_n, the expected number of steps it takes to get Home, which we take to be at the point 0, while we have the drunkard or frog start from a position n. We may find that $a_n = 1 + \frac{a_{n-1}+a_{n+1}}{2}$, so $a_{n+1} = 2a_n - a_{n-1} - 2, q(x) = 1 - 2x + x^2 = (1-x)^2$.

Take a generating function $G(x) = \sum_n a_n x^n$ and compute $G(x)(1-x)^2$ which we can expand to recognize as $\frac{-2}{1-x}$.

6 Thursday, April 14, 2016

SUBSTITUTE: Prof. Razborov

Recap: Take a generating function $G(x)$ such that $G(x)(1-x^2) = ?+? - 2x^2 - 2x^3 - \ldots = L(x) - \frac{2}{1-x}$. So, we write $G(x) = \frac{L(x)}{(1-x)^2} - \frac{2}{(1-x)^3} = \frac{p(x)}{(1-x)^3}$, and note $\frac{1}{(1-x)^3} \approx + \ldots + \binom{n}{3}x^n + \ldots$ Meanwhile, our recurrence relation also gives $a_{n+2} = 3a_{n+1} - 3a_n + a_{n-1}$. Moreover, we see that our steps needed to get back home converges.

Binary trees: n nodes, $b_n = \#\text{trees}$, and we count the number of branches that go leftward from a node l and those that go rightward k. Then $k + l = n - 1$. Also, we obtain the recurrence relation $b_n = b_0 b_{n-1} + b_1 b_{n-2} + \ldots + b_{n-1} b_0$.

We can represent this relation with a generating function so that $1 + xG(x)^2 = G(x)$, which has solutions $G_1(x) = \frac{1+\sqrt{1-4x}}{2x}$ and $G_2(x) = \frac{1-\sqrt{1-4x}}{2x}$, but only G_2 is still analytic at $x = 0$, so we choose this one as our generating function.

Meanwhile, we can write $(1-4x)^{1/2} = \sum_{k=0}^{\infty} (-4)^k \binom{1/2}{k} x^k$. Here, the coefficients are $\binom{1/2}{k} = \frac{(1/2)(-1/2)(-3/2)\cdots}{k!}$ and thus we obtain $b_n = \binom{2n}{n}/(n+1)$, **Catalan numbers**.

Now, consider the **Young tableau** If we have $i_1, \ldots, i_k$ squares in rows $1, \ldots, k$, respectively, then we can count the total number of representations of the table $i_1 + 2i_2 + 3i_3 + \ldots + ki_k = n$. Equivalently, we can write $x^n = x^{i_1} x^{2i_2} \cdots x^{ki_k} = x^{i_1}(x^2)^{i_2} \ldots (x^k)^{i_k}$. Then we can write a generating function

$$
\begin{aligned}
G(x) &= \textstyle\sum_n p_n x^n \\
&= (1 + x + x^2 + \ldots + x^n + \ldots)(1 + x^2 + x^4 + x^6 + \ldots)(1 + x^3 + x^6 + x^9 + \ldots)\cdots \\
&= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots\frac{1}{1-x^k}\cdots,
\end{aligned}
$$

i.e. $x^n p_n \leq G(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}, x \in (0,1)$, where $p_n \leq \frac{1}{x^n} \prod_{k=1}^{\infty} \frac{1}{1-x^k}$. This implies that $\ln p_n \leq -n \ln x - \sum_{k=1}^{\infty} \ln(1-x^k)$. Note $\ln(1-x) = \sum_j \frac{x^j}{j}$ and $\ln(1-x^k) = \sum_j \frac{x^{jk}}{j}$. We need to sum over such terms: $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{x^{jk}}{j} = \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=1}^{\infty} x^{jk}$. So, we obtain $\ln(1-x^k) = \sum_{j=1}^{\infty} \frac{1}{j} \frac{x^j}{1-x^j}$.

Meanwhile, we have $1 - x^j \geq j(1-x)x^{j-1} = jx^{j-1} - jx^j$, since $(j-1)x^j \geq jx^{j-1} - 1$. So, $\ln(1-x^k) \leq \frac{x}{1-x} \sum_{j=1}^{\infty} \frac{1}{j^2}$. Thus, $\ln p_n \leq -n \ln x - \frac{x}{1-x} = \frac{x}{1-x}\frac{\pi^2}{6}$.

Kruskal-Katona Theorem: $\mathcal{F} \subseteq \binom{[n]}{k}, |\mathcal{F}| = m, \partial\mathcal{F} \subseteq \binom{[n]}{k-1} = \{G \in \binom{[n]}{k-1} \mid \exists F \in \mathcal{F} \text{ such that } G \subseteq \bar{F}\}$. How small can $\partial\mathcal{F}$ be? If $m = \binom{x}{k}$, where $x \geq k$, then $|\partial\mathcal{F}| = \binom{x}{k-1} = \frac{x(x-1)\cdots(x-k+1)}{k!}$.

The theorem in **Lovasz's form**: For every $m = \binom{x}{k}, |\partial\mathcal{F}| \geq \binom{x}{k-1}$.

7 Tuesday, April 19, 2016

For hypergraphs without multiple edges $\mathcal{H} = (V, \mathcal{E})$ and $\mathcal{H}' = (V', \mathcal{E}')$, an **isomorphism** is a bijection $f : V \to V'$ such that $\forall F \subseteq V, F \in \mathcal{E} \iff f(F) \in \mathcal{E}'$. Likewise, $\mathcal{H}$ and $\mathcal{H}'$ are **isomorphic**, i.e. $\mathcal{H} \cong \mathcal{H}'$, if $\exists f : \mathcal{H} \to \mathcal{H}'$ isomorphism.

An **automorphism** of $\mathcal{H}$ is a function $f : \mathcal{H} \to \mathcal{H}$ that is an isomorphism. Equivalently, it is a permutation of V.

DO: #automorphisms of Fano = #hours in a week. Note that this is the 2^{nd} smallest nonabelian simple group. (The smallest is A_5, since $|A_5| = 60$.)

S_n : **group** of permutations of $[n]$, symmetric of degree n. $|S_n| = n!$ is the order of the symmetric group. A subgroup $G \leq S_n$ is a **permutation group of degree** n

G is **transitive** if $\forall i, j \in [n], \exists \sigma \in G$ such that $\sigma(i) = j$.

$\mathcal{H}$ is **vertex-transitive** if $\text{Aut}(\mathcal{H})$, the **automorphism group**, is transitive.

DO: Fano plane is vertex-transitive, as are all other Galois planes.

DO: SET_d is vertex-transitive.

Observation: vertex-transitive $\implies$ regular

DO: regular $\not\Longrightarrow$ vertex-transitive. (Find smallest graph with $k = 2$.)

DO: Platonic solids, as graphs, are vertex-transitive, edge-transitive, face-transitive, and in fact flag-transitive

Unrelatedly, note from an earlier HW problem: $\chi(SET_d) \to \infty$ as $d \to \infty$ due to the best lower bound on optimal coloring χ: $\chi \geq n/\alpha \geq 3^d/(2 \cdot 3^d/d) = d/2$, using Meshulam's Theorem.

Also, note events A and B are **disjoint** if $P(A \cap B) = 0$.

DO: Union bound yields equality $\iff$ the events are pairwise disjoint.

This fact is important to note while showing before that

$$P(\text{illegal coloring}) \leq \sum_{E \in \mathcal{E}} P(E \text{ monochromatic}) = m/2^{k-1} \leq 1$$

DO: If a projective plane has order ≥ 3, then $\chi = 2$.

Also, returning to the drunkard's problem: note that, if the drunkard takes n steps, then, in order to get home (initial position) n must be even, $P(\text{getting home at the } 2n^{\text{th}} \text{ step}) = \binom{2n}{n}/2^{2n}$.

Also, we can find that a walk (x-y graph of location vs. time) will cross the x axis a certain number of times. Each segment between closest times at which the drunkard reaches the x axis will have a twin path could have been taken in that give walk, i.e. the same path reflected about the x axis. This **reflection principle** implies that, instead of counting $\binom{2n-2}{n-1}$ non-crossing walks, we can neglect $\binom{2n-2}{n}$ walks. More precisely,

$$\#\textbf{non-crossing walks} = \frac{\binom{2n-2}{n-1} - \binom{2n-2}{n}}{\binom{2n-2}{n-1}} = 1 - \frac{\binom{2n-2}{n}}{\binom{2n-2}{n-1}} = 1 - \frac{n-1}{n} = \frac{1}{n}.$$

Thus, we find the Catalan numbers $\binom{2n-2}{n-1}/n$. (We can shift using $n = n' + 1$ to obtain the numbers' usual formula.)

DO: $\binom{2n}{n}/2^{2n} \sim 1/\sqrt{\pi n}$ (Note $2^{2n} = \sum_{k=0}^{2n} \binom{2n}{k}$.)

$P(\text{getting back in } 2n \text{ steps for first time}) = \frac{1}{n}\binom{2n}{n}/2^{2n} \sim 1/\sqrt{\pi n^3}$

HW: $E(\#\text{steps for drunkard to get home for first time})$ asymptotically. [You can use Stirling's formula.]

8 Thursday, April 21, 2016

Erdös-Szekeres: $\forall a_1, a_r, a_{kl+1} \in \mathbb{R}$,

(i) $\exists$ increasing subsequence of length $k+1$
or (ii) $\exists$ non-increasing subsequence of length $l+1$

Need kl pigeon holes: $\{(i,j) \mid 1 \le i \le k, 1 \le j \le l\}$. Assume (i) and (ii) fail. Consider a map of indices to pigeonholes $r \mapsto (i,j)$. We want i = length of longest increasing subsequence ending in a_r and j = length of largest non-increasing subsequence ending in a_r. Suppose we map $a_r \mapsto (i,j)$ and $a_s \mapsto (i',j')$.

Case 1: $a_r < a_s$. Then $i' \ge i+1$.
Case 2: $a_r \ge a_s$. Then $j' \ge j+1$.

Erdös took Prof. Babai, at 16, to his mother's home for lunch once, as Babai was amongst the "epsilons" of rising mathematicians. He asked Babai a first test question: $A \subseteq \{1, \dots, 2n\}$; what is the smallest size of A such that it certainly has two consecutive elements? Incidentally, the prodigy P'osa was an epsilon in 7th grade and solved the same question quickly, only hesitating for a moment after raising his spoon while eating. There was a second problem Erdös would ask, whose answer was that one of the elements of A divides another.

$\mathcal{H} = \{V; A_1, \dots, A_m\}, A_i \subseteq V$. If $i \ne j$, then $|A_i \cap A_j| = 1$. Find $\max m$ as a function of n.

Easy attempts: Consider edges that are 2-sets all with one vertex of mutual intersection, giving $n-1$ edges; we can then add an edge containing only that intersection or an edge containing all vertices but that intersection, yielding a total of n edges.

Alternatively, one could find using a map to a finite projective plane to also obtain an $m = k^2 + k + 1$.

Erdös-deBruijn: $m \leq n$.

Generalized Fisher inequality: Fix $\lambda \geq 1$. $|A_i \cap A_j| = \lambda$. Then, $m \leq n$. (Fisher $\sim$ 1930 in the Journal of Eugenics [for plants]. This inequality for uniform hypergraphs, which is more general than this result, was found by R.C. Bose in 1949 and Majumdar in 1955.)

Proof: Incidence matrix: $M = (m_{ij}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ & 1 & & \\ & & & \end{pmatrix}$, $m_{ij} = \begin{cases} 1 \text{ if } j \in A_i \\ 0 \text{ if } j \notin A_i \end{cases}$. $A_i \mapsto$ v_i : incidence vector of A_i, $v_i \in \mathbb{R}^n$.

Claim: Under the conditions of the theorem, $v_1, \ldots, v_m$ are linearly independent. $\therefore m \leq n$

$A, B \subseteq V$, define dot product: $x, y \in \mathbb{R}^n, \underline{x} \cdot \underline{y} = \sum x_i y_i$

$v_A \cdot v_B = |A \cap B|$

$v_A \cdot v_A = |A \cap A| = |A|$

Thm*: $v_1, \ldots, v_m \in \mathbb{R}^n, v_i \cdot v_i > \lambda, v_i \cdot v_j = \lambda, (i \neq j)$. Then, the v_i are linearly independent.

Case 2: $\forall i, |A_i| > \lambda$

Case 1: $\exists A_i, |A_i| = \lambda$

Sunflower: $A_1, \ldots, A_m$ such that $\exists k$ "kernel": $\forall i, K \subseteq A_i$ and $\forall i \neq j, A_i \cap A_j = K$

Need to show: $\forall \alpha_1, \ldots, \alpha_m \in \mathbb{R}$, if $\sum \alpha_i v_i = 0$, then $\alpha_1 = \ldots = \alpha_m = 0$

$0 = (\sum_{i=1}^m \alpha_i v_i) \cdot v_j = \sum_{i=1}^m \alpha_i (v_i \cdot v_j) = \lambda \sum_{i=1}^m \alpha_i + \alpha_j (v_j \cdot v_j - \lambda) = \lambda T + \alpha_j (k_j - \lambda)$, where $T = \sum_{i=1}^m \alpha_i, |A_j| = k_j$

So, $\alpha_j = \frac{-\lambda T}{k_j - \lambda}$

$T = \sum \alpha_j = -\lambda T \sum \frac{1}{k_j - \lambda}$

$\implies 0 = T(1 + \lambda \sum \frac{1}{k_j - \lambda})$

$\implies T = 0$ b/c other term is positive $\implies \alpha_j = 0$ b/c $\alpha_j = \frac{-\lambda}{k_j - \lambda} T$

Now, consider Clubtown, where there are $V = \{\text{citizens}\}$, clubs $A_1, \ldots, A_m$, $A_i \neq A_j, |A_i|$ even. In Eventown, $|A_i \cap A_j|$ even

DO: Find $2^{\lfloor \frac{n}{2} \rfloor}$ clubs

REWARD: $2^{\lfloor \frac{n}{2} \rfloor}$ is maximum

$m \leq 2^n$

DO: $m \leq 2^{n-1}$ (max #even subsets = 2^{n-1})

CH: Every maximal Eventown system is maximal.

9 Tuesday, April 26, 2016

Consider the problem: Given $\alpha, \beta \geq 0$; $\forall i \neq j, |A_i \cap A_j| \in \{\alpha, \beta\}$ find an example of $\mathcal{H}$ such that $m = n$

All 2-element sets $\binom{n}{2}$ sets with intersection sizes $L = \{0, 1\}$
All sets of size ≤ 2: $\binom{n}{2} + n + 1 = \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$.
s intersections: $L = \{l_1, \ldots, l_s\}$
All s-subsets: $\binom{n}{s}$ uniform
All sets of size $\leq s$:
$\binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{1} + \binom{n}{0}$ non-uniform.

Ray-Chaudhuri-Wilson Theorem (1964): If $\mathcal{H}$ is uniform and $s \leq n/$ and L-intersecting then $m \leq \binom{n}{s}$, where $s = |L|$. ($\mathcal{H}$ is L-intersecting if $\forall i \neq j, |A_i \cap A_j| \in L$.)

Non-uniform RW Theorem (Frankl-Wilson (1980)): If $\mathcal{H}$ is L-intersecting, then $m \leq \binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{1} + \binom{n}{0}$

Proof [of "non-uniform"] (Babai 1988): $\underline{x}, \underline{y} \in \mathcal{R}^n$ $f(\underline{x}, \underline{y}) = \prod_{i=1}^{s} (\underline{x} \cdot \underline{y} - l_i)$.

v_i is an incidence vector of A_i
for $j \neq k$, $f(v_j, v_k) = \prod_{i=1}^{s} (v_j \cdot v_i - l_i) = 0$, where $\underline{v}_j \cdot \underline{v}_k = |A_j \cap A_k|$
for $j = k$, $f(v_j, v_j) \neq 0$?
No: if $|A_j| \in L$, then we get 0.
Suppose $f(v_j, v_k) = 0 \iff j \neq k$.
$f_j(x) = f(\underline{x}, \underline{v}_j)$, m polynomials in n variables each
Claim: then $f_1, \ldots, f_m$ are linearly independent.
$\mathcal{P}(n, s)$ = space of polynomials of degree $\leq s$ in variables $x_1, \ldots, x_n$
$\alpha_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n + \sum_{i<j} \alpha_{ij} x_i x_j + \sum \beta_j x_j^2$, a polynomial with
$n + 1 + \binom{n}{2} + n$ terms.

DO: general case: $\binom{n}{s}(1 + o(1))$ (i.e. $\binom{n}{s}(1 + a_n)$ where $a_n \to 0$ as $n \to \infty$)

HW: Find exact dim, a closed-form expression–very simple. Need to count: monomials of deg $\leq s$, $\deg(x_1^3 x_2 x_5 x_7^2) = 1$.

Proof of Claim: Suppose $\gamma_1 f_1 + \ldots + \gamma_m f_m = 0$
Need to show: $\gamma_1 = \ldots = \gamma_m = 0$
$0 = \sum_j \gamma_j f_j(v_k) = \gamma_k f_k(v_k), \therefore \gamma_k = 0$
($f_j(v_k) = 0$ if and only if $k = j$)

HW: Suppose $f(v_j, v_k) = \begin{cases} \neq 0 \text{ if } j = k \\ 0 \text{ if } j < k \end{cases}$ (i.e., triangular condition). (a) Prove the f_j are linearly independent. (b) The multilinearization of polynomials: $x_1^3 x_2 x_5 x_7^2 \mapsto x_1 x_2 x_5 x_7$; $f \mapsto \tilde{f}$. Show the $\tilde{f}$ are linearly independent. (Remember: the v_j are $(0,1)$-vectors)

THINK: Find an ordering of the A_i and fix the definition of f_j such that the triangular condition is true.

#multilinear polynomials of degree s : $\binom{n}{s}$ count multilinear monomials of degree $\binom{n}{s}$

Tuesday, May 3, 2016

Reviewing Quiz 2, Problem 1:

(a) find max r such that $\forall x, 0 < x < r, a_n x^n \to 0$
  $a_n \approx 4^n$, intuitively, but more precisely: $a_n \sim 4^n/\sqrt{\pi n}$ by Stirling's

DO: $\lim_{n\to\infty} a_{n+1}/a_n = 4$ – verify directly (no Stirling's)

  Note $\sum_{i=0}^{2n} \binom{2n}{i} = 2^{2n} = 4^n$, while $\binom{2n}{n} > 4^n/(2n+1)$ obviously $> 3.9^n$

(b) Convergence radius $r = 1/4$: $f(x) = \sum_{n=0}^{\infty} a_n x^n = 1/\sqrt{1-4x}$
  $1/(1-4x)^\alpha = \sum \binom{-1/2}{n} 4^n x^n$
  Computing $\binom{-1/2}{n}$ from a DO problem into ordinary binomial coefficients
  Then, we'd find we need to require $\alpha = -1/2$.

Problem 2:

  Taking $2n$ steps, total # walks that start at 0 and end at 0 in $2n$ steps: $\binom{2n}{n}$
  $\# = \binom{2n}{n-k} = \binom{2n}{n+k}$

Problem 4:

  $(X_1, \dots, X_n)$: n integers $1 \le X_i \le 6$
    Sample space: $|\Omega| = 6^n$
    $E(\sum X_i) = \sum E(X_i) = 7n/2$
    $\text{Var}(\sum X_i) = \sum \text{Var}(X_i) = n\text{Var}(X_i)$ b/c X_i pairwise independent
    $E(\prod X_i) = \prod E(X_i) = (7/2)^n$ b/c X_i fully/mutually independent

Bonus:

  $b_n = n^2, b_n = \alpha b_{n-1} + \beta b_{n-2} + \gamma b_{n-3}$ (homogeneous 3rd order recurrence)
    $c_n = \Delta b_n = b_n - b_{n-1}$=linear=$2n-1$
    $d_n = \Delta c_n = c_n - c_{n-1}$=constant=2
    $\Delta d_n = 0$

$n^2 - 3(n-1)^2 + 3(n-2)^2 - (n-3)^2 = 0$: $1 - 3 + 3 - 1$

HW problems discussed:

$\mathcal{Q}(n, 5)$: space of homogeneous polynomials of degree s in n variables
$\mathcal{P}(n, 5)$: space of homogeneous polynomials of degree s in $\leq n$ variables
Here, we have the equation $k_1 + \ldots + k_n = 5$, $k_i \geq 0$
from which we have to count #solutions in unknowns $k_1, \ldots, k_n$
Using a stars and bars approach of counting:
we would have s stars and $n-1$ bars, to obtain $s+n-1$ binary symbols
result: $\binom{n+s-1}{s}$
Then, $\dim \mathcal{Q}(n,s) = \binom{n+s-1}{s}$ and
$\dim \mathcal{P}(n,s) = \sum_{j \leq s} \dim \mathcal{Q}(n,j) = \binom{n+s}{s} = \dim \mathcal{Q}(n+1, s)$
Claim: $\dim P(n,s) = \dim Q(n+1, s)$

DO: $\sum_{j=0}^{s} \binom{n+j-1}{j} = \binom{n+s}{s}$

Recall the definition/notation for $f \mapsto \tilde{f}$ **multilinearization** (e.g., $x_1^3 x_4 x_5^2 \mapsto x_1 x_4 x_5$)

Suppose $\Omega = \mathbb{R}^n$, f_i polynomials, v_i=(0,1)-vectors$\in \{0,1\}^n$ $\therefore$ $f_i(v_i) = \tilde{f}_i(v_i)$.

Setup: $f_1, \ldots, f_m$ functions over domain Ω, $v_1, \ldots, v_m \in \Omega$, $f_i(v_j) = \begin{cases} \neq 0 & i = j \\ 0 & i < j \end{cases} \implies$ $f_1, \ldots, f_m$ are linearly independent. (Proof uses induction on m; $m = 1$: $\checkmark, m \geq 2$: Suppose $\sum \alpha_i f_i = 0$. Claim $\alpha_m = 0$. Plug in v_m : $\sum \alpha_i f_i(v_m) = \alpha_m f_m(v_m) = 0 \implies \alpha_m = 0$ since $f_i(v_m) = 0$ unless $i = m$ and $f_m(v_m) \neq 0$.)

Frankl-Wilson Theorem (Non-uniform version: **Ray-Chandhuri-Wilson Theorem**): If $A_1, \ldots, A_m \subseteq [n], L = \{l_1, \ldots, l_s\}, \forall i \neq j |A_i \cap A_j| \in L$, then $m \leq \binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{0}$, by considering all subsets of size $|A_i| \leq s$, tight for all n and all $s \leq n$, $L = \{0, \ldots, s-1\}$

$|A_1| \geq \ldots \geq |A_m|$ and take v_i as incidence vectors of A_i
$f_i(\underline{x}) = \prod_{j: l_j < |A_i|} (\underline{v}_i \cdot \underline{x} - l_j)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$ polynomials in n variables.
$f_i(v_i) = \prod_{l_j < |A_i|} (|A_i| - l_j) \neq 0$, $f_i(v_j) = \prod_{k, l_j < |A_i|} (|A_i \cap A_j| - l_k)$
Claim: $|A_i \cap A_j| < |A_i|$ (Proof: o/w, if $A_i \subset A_j \implies |A_i| < |A_j| \implies \Longleftarrow$)

Ray-Chandhuri-Wilson Theorem: If additionally uniform $|A_1| = \ldots = |A_m| = k$, then $m \leq \binom{n}{s}$ assuming $s < n/2$, tight $\forall n, \forall s < n/2$, (all subsets of of size s)

Proof: We know that $\tilde{f}_1, \ldots, \tilde{f}_m$ linear independence. Q is the space of multilinear polynomials of deg $\leq s-1$, $\dim Q = \binom{n}{s-1} + \binom{n}{s-2} + \ldots + \binom{n}{0} = \#g_I$; for $I \subseteq [n], |I| \leq s-1, g_I = \prod_{i \in I}(x_i - 1)$.

HW [for next Tues]: Claim: All the g_I and $\tilde{f}_i$ are linearly independent. $\therefore \dim Q + m \leq \binom{n}{s} + \binom{n}{s-1} + \ldots \binom{n}{0}$, i.e. $m \leq \binom{n}{s}$

$k \times n$ **Latin rectangle**: $k \le n$, $k \times n$ matrix $a_{ij} \in \{1, \ldots, n\}$, every row and column has at most 1 occurrence of each value

Graph: **matching** is a set of disjoint edges; **perfect matching** has $n/2$ edges too

Theorem: Non-empty regular bipartite graph always has a perfect matching.

HW: Use this to prove: any Latin rectangle can be completed to a Latin square

Determinant: $M_n(\mathbb{R}) \to \mathbb{R}, M_n(\mathbb{R}) = \{n \times n \text{ matrices}\}, A \in M_n(\mathbb{R}), A = (a_{ij}), \det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$

Permanent: $\operatorname{per}(A) = \sum_\sigma \prod_{i=1}^n a_{i\sigma(i)}$ (e.g., $\operatorname{per}(\mathbb{I}) = 1$, $\operatorname{per}(J) = n!$ [J is the all-ones matrix], $\operatorname{per}(\frac{1}{n}J) = n!/n^n > e^{-n}$

Note $A \to A'$ times λ, then $\operatorname{per}(A') = \lambda \operatorname{per}(A)$ and $\operatorname{per}(\lambda A) = \lambda^n \operatorname{per}(A)$

A is **stochastic** if every row is a probability distribution, i.e. $\forall a_{ij} \ge 0, \forall i, \sum_j a_{ij} = 1$

A is **doubly stochastic** if both A and A^T are stochastic, i.e. $\forall j, \sum_i a_{ij} = 1$.

DO: If A is stochastic, then $\operatorname{per}(A) \le 1$

DO: prove $n!/n^n > e^{-n}, \forall n$ [1-line proof, no Stirling's]

The Permanent Inequality, (Egorychev & Falikman): If A doubly stochastic, then $\operatorname{per}(A) \ge n!/n^n$ (used to be called van der Waerchen's conjecture) [Proof in van Lint-Wilson]

10 Thursday, May 5, 2016

HW problem discussed:

$S(n,k) = \sum_{t=0}^{\infty} \binom{n}{kt} \stackrel{?}{=} \frac{1}{k} \sum_{j=0}^{k-1} (1+\zeta^j)^n$
where $\zeta = e^{2\pi i/k}$ is (the first) k^{th} root of unity (along the unit circle)
$= \frac{1}{k} \sum_{j=0}^{k-1} \sum_{l=0}^{n} \binom{n}{l} \zeta^{jl} = \frac{1}{k} \sum_{l=0}^{n} \binom{n}{l} \sum_{j=0}^{k-1} \zeta^{jl}$

DO: Powers of ζ^l are all k^{th} **roots of unity** ($x^k = 1, x \in \mathbb{C}$) $\iff$ $\gcd(k,l) = 1$ (relatively prime)

DO: ζ^l is an m^{th} root of unity where $m = k/\gcd(k,l)$.

Note if $k|l$, then $\zeta^l = 1$
The **order** of $z \in \mathbb{C}$ is the smallest $m \ge 1$ such that $z^m = 1$
$\exists$order $\iff$ z is a root of unity
If $\operatorname{ord}(z) = m$, then we say that z is a **primitive** m^{th} **root of unity**

DO: Suppose z is a root of unity. $z^s = 1 \iff \text{ord}(z)|s$

$a|b$ (i.e., a divides b) if $\exists x$ such that $ax = b$. Note $0|0$.

d is a **greatest common divisor (gcd)** if $\underline{a}$ and $\underline{b}$ if (a) $\underline{d}$ is a common divisor (i.e., $d|a$ and $d|b$) and (b) $\forall e$, if $e|a$ and $e|b$ (i.e., $\underline{e}$ is a common divisor), then $e|d$

DO: $\forall x, a|x \iff a = \pm 1$

DO: Understand $\gcd(0,0) = 0$

DO: $\forall a, b$, if d is a gcd, then $d, -d$ are the only gcds

Note if d is a $\gcd(a,b)$ then $-d$ is also a $\gcd(a,b)$. Convention: $\gcd(a,b)$ denotes the non-negative gcd

DO: $\text{ord}(z^l) = \text{ord}(z) \iff \gcd(\text{ord}(z), l)$

DO: $\text{ord}(z^l) = \text{ord}(z)/\gcd(l, \text{ord}(z))$

Suppose $\text{ord}(z) = k$. Then powers of z^l are exactly the m^{th} roots of unity, $m = k/\gcd(k,l)$ $z^{lj}, j = 0, \ldots, k-1 \implies$ each m^{th} root of unity occurs k/m times.

DO: If z is a k^{th} root of unity $z \neq 1$, then $\sum_{j=0}^{k-1} z^j = 0$

Thus, $S(n,k) = \frac{1}{k} \sum_{l=0,k|l} \binom{n}{l} k$,
since $\sum_{j=0}^{k-1} \zeta^{jl} = 0$ unless $\zeta^l = 1 \iff k|l$.

Then, $S(n,2) = \binom{n}{0} + \binom{n}{2} + \ldots = 2^{n-1} = 2^n/2$, if $n \geq 1$, while $S(0,2) = 1 = 2^0 \neq 2^0/2$.

If $0^n = \begin{cases} \text{if } n = 0, & 1 \\ \text{if } n \geq 1, & 0 \end{cases} = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots$

HW: $|S(n,3) - 2^n/3| < 1$. Use formula for $S(n,3)$.

Sunflower: set system $A_1, \ldots, A_m$ such that $K := \cap_{i=1}^m A_i$. Then the set $A_i - K$ are disjoint **petals**, i.e., $\forall i \neq j, A_i \cap A_j = K$.

HW: Suppose $\forall i, |A_i| \leq r$, the A_i are distinct. Suppose $m > (s-1)^r r!$. Then, $\exists$sunflower with s petals.

Matching in a hypergraph $\mathcal{H}$: a set of disjoint edges, $\nu(\mathcal{H}) = \max \#$disjoint edges **matching number**

If $\mathcal{H} = (V, \mathcal{E})$, a **cover** or "**hitting set**" of $\mathcal{H}$ is a subset $W \subseteq V$ such that $\forall E \in \mathcal{E}, W \cap E \neq \emptyset$. $\tau(\mathcal{H}) = $ min size of a hitting set "**covering number**"

HW: (a) $\nu(\mathcal{H}) \le \tau(\mathcal{H})$ [direct proof, w/o anything fractional], (b) If $\mathcal{H}$ is r-uniform, then $\tau(\mathcal{H}) \le r\nu(\mathcal{H})$, (c) $\forall r \ge 2, \forall \nu \ge 1$, show: both (a) and (b) are tight.

Fractional cover: $f : V \to \mathbb{R}$, $\forall v \in V, f(v) \ge 0; \forall E \in \mathcal{E}, \sum_{v \in E} f(v) \ge 1$.

Integer solutions to this system of $n+m$ inequalities gives a cover of size $\sum_{v \in V} f(v)$ (value of fractional cover)

$\tau^*(\mathcal{H}) = \text{fractional cover number} = \min_{f:\text{fractional cover}} \text{value}(f)$

DO: $\tau^* \le \tau$

Fractional matching: $g : \mathcal{E} \to \mathbb{R}$ such that $\forall E \in \mathcal{E} g(E) \ge 0$ and $\forall v \in V \sum_{E, v \in E} g(E) \ge 1$. value$(g) = \sum_{E \in \mathcal{E}} g(E), \nu^*(\mathcal{H}) = \max_{g:\text{fractional matching}} \text{value}(g)$

HW: $\nu^* \le \tau^*$

Corollary: $\nu \le \nu^* \le \tau^* \le \tau$

HW: Find ν, τ, ν^*, τ^* for all finite project planes (in terms of the order). [Do not use the result below.]

Theorem: $\nu^* = \tau^*$ by the **Linear Programming Duality Theorem**

DO: C_n, n-cycles: $\tau(C_n) = \lceil n/2 \rceil$. $\nu(C_n) = \lfloor n/2 \rfloor$. $\tau^* \le n/2$, $\nu^* \ge n/2$ (assign 1/2 as weight for each of the n points). So, $n/2 \le \nu^* \le \tau^* \le n/2$.

11 Tuesday, May 10, 2016

$\forall$hypergraph $\mathcal{H} = (V, \mathcal{E})$, $\alpha\chi \ge n$.

> Pf: every color class is independent
> If $\mathcal{H}$ is a vertex-transitive hypergraph ($\forall v_1, v_2 \in V, \exists \pi \in \text{Aut}(\mathcal{H})$ such that $\pi(v_1) = v_2$, then $\alpha\chi \le n(1 + \ln n)$.

DO: χ is the min #independent sets of which the union is V

> Pf: Let A be an independent set of size α, $C(v) := \min\{i \mid v \in C_i\}$
> $G = \text{Aut}(\mathcal{H})$. Pick $\pi_1, \dots, \pi_s \in G$ uniformly, independently at random

DO: If $\mathbb{P}(\cup_{i=1}^{s} \pi_i(A) \ne V) < 1$, then $\chi \le s$. [Need to show the inequality for as small an n as we can do.]

DO: Fix a vertex v. $\mathbb{P}(v \notin \pi_1(A)) = 1 - \alpha/n$. This is where we use vertex-transitivity. also, this proof does not depend on the vertex.

$\mathbb{P}(v \notin \pi_2(A))$ =same
$\mathbb{P}(v \notin \pi_1(A) \cup \ldots \cup \pi_s(A)) = (1-\alpha/n)^s$
$\mathbb{P}((v \notin \pi_1(A)) \cap \ldots \cap (v \notin \pi_s(A))) = \prod \mathbb{P}(v\pi_i(A))$
b/c the indicator variables of the events are functions of the independent random variables π_i.

Cor (so far): if $n(1-\alpha/n) < 1$, then $chi \le s$.

$\mathbb{P}(\exists v \notin \cup_{i=1}^s \pi_i(A)) = \mathbb{P}(\cup_{v\in V}(v \notin \cup_{i-1}^s \pi_i(A))) \le \sum_{v\in V} \mathbb{P}(v \in \cup \ldots) = n(1-\alpha/n)^s$

DO: $\forall x \ne 0, 1+x < e^x$, it suffices $ne^{-\alpha s/n} \le 1$, $n \le e^{\alpha s/n}$.

Then, $1-\alpha/n < e^{-\alpha/n}$, so $(1-\alpha/n)^s < e^{-\alpha s/n} \implies \ln n \le \alpha s/n \implies s \ge n \ln n/\alpha$. So, $s := \lceil n \ln n/\alpha \rceil$.

Old HW:

$\nu(\mathcal{H})$ matching #: max size of matching = max#disjoint edges
$\tau(\mathcal{H})$ covering/hitting/transversal #: min size of cover
[cover/hitting/transversal subset $S \subset V$ such that $\forall E \in \mathcal{E}, E \cap S \ne \emptyset$
If $\mathcal{H}$ is k-uniform, then $\tau \le k\nu$. $T = \cup_{i=1}^s E_i$, T is a cover.
Maximal matching $s \le \nu$.
$\nu \le \tau \le k\nu$, $\forall k \ge 2, \nu \ge 1$ ($\mathcal{H}$: ν disjoint edges)
Case $\nu = 1$: find intersecting k-uniform hypergraph with $\tau = k$.
$n = 2k-1$, $\mathcal{E} = \binom{[n]}{k}$ set of k-tuples, complete k-uniform hypergraph on $2k-1$ vertices
So far, we have $\nu \le \nu^*, \tau^* \le \tau$. Claim: $\nu^* \le \tau^*$
Need to show $\forall$fractional matching g and $\forall$fractional covering f, $\text{val}(g) \le \text{val}(g)$.
We have $S := \sum_{(v,E), v\in E} f(v)g(E)$
$S = \sum_{v\in V} \sum_{E, v\in E} f(v)g(E) = \sum_{v\in V} f(v) \sum_{E,v\in E} g(E) \le \sum_{v\in V} f(v) = \text{val}(f)$
while $S = \sum_{E\in\mathcal{E}} \sum_{v\in E} f(v)g(E) = \sum_{E\in\mathcal{E}} g(E) \sum_{v\in E} f(v) \ge \sum_{E\in\mathcal{E}} g(E) = \text{val}(g)$
Thus, $\text{val}(g) \le S \le \text{val}(f)$. (QED)

DO: If $\mathcal{H}$ is k-uniform, then $\nu \le n/k$.

DO: T/F? If $\mathcal{H}$ is k-uniform, then $\nu^* \le n/k$.

DO: If $\mathcal{H}$ is k-uniform and regular, then $\nu^* = \tau^* = n/k$. [Do not use the Duality Theorem.]

DO: If $\mathcal{H}$ is k-uniform, then $\tau \le \lceil n \ln m/k \rceil$

DO: Use this to prove $\alpha\chi \le \cdots$ for vertex-transitive hypergraphs.

A **Sperner family** is a set system $A_1, \ldots, A_m$ such that no two are **comparable** ($\forall i \ne j, A_i \not\subseteq A_j$).

Sperner's Theorem: $m \le \binom{n}{\lfloor n/2 \rfloor}$. (All k-subsets: $m = \binom{n}{k}$, $k = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$.)

Lemma: If $A_1, \ldots, A_m \subseteq [n]$, Sperner family then $\sum_{i=1}^m 1/\binom{n}{|A_i|} \leq 1$

DO: (a) Use this to prove theorem. (b) Find Sperner families for which equality holds in the Lemma.

CH: Find all such families.

12 Thursday, May 12, 2016

BLYM Inequality: $A_1, \ldots, A_m \subseteq [n]$, Sperner $\implies \sum_{i=1}^m 1/\binom{1}{|\Delta_i|} \leq 1$. Equality occurs when complete k-uniform, $\forall k$.

Q: Are there any other cases?

Note, if $A_1, \ldots, A_m$ are all the subsets and $\forall i \neq j, A_i \not\subseteq A_j$, then $\sum_{A \subseteq [n]} 1/\binom{n}{|A|} = n + 1$

Then, Sperner's Thoerem (if Sperner family, then $m \leq \binom{n}{\lfloor n/2 \rfloor}$, equality if complete k-uniform with $k = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. [Pf from BYLM: $m \leq \sum_{i=1}^m \binom{n}{\lfloor n/2 \rfloor}/\binom{n}{|A_i|} \leq \binom{n}{\lfloor n/2 \rfloor}$, noting $\binom{n}{\lfloor n/2 \rfloor}/\binom{n}{|A_i|} \geq 1$.]

DO$^+$: These are the only cases of equality.

Recall Lubell's Permutation Method. σ ordering of $[n]$, $A \subseteq [n]$ [EKR was proved with cyclic permutation method by Katona.] (A, σ) are compatible if A is a **prefix** under σ (i.e., sequence of adjacent elements in the given ordering)

σ: random linear ordering, $|\Omega| = n!$, $A_1, \ldots, A_m$ Sperner family, $N(\sigma) = \#i$ such that A_i is compatible with σ.

(1) $\forall \sigma, N(\sigma) \leq 1 \impliedby$ Any two prefixes of a linear order are compatible.
(2) $1 \geq E(N(\sigma)) = \sum E(X_i) = \sum \mathbb{P}(A \text{ is compatible with } \sigma) = \sum 1/\binom{n}{|A_i|}$
$N(\sigma) = \sum_{i=1}^m X_i$, X_i indicator of "A_i is compatible with σ"
$\mathbb{P}(A_i \text{ compatible with } \sigma) = |A_i|!(n - |A_i|)!/n! = 1/\binom{n}{|A_i|}$ [over choice of σ]

HW [next Thurs]: (**baby Littlewood-Offord**) Given $a_1, \ldots, a_n, b \in \mathbb{R}, a_i \neq 0$, take $I \subseteq [n]$ at random, and note we have $|\Omega| = 2^n$. Prove $\exists c$ such that $\mathbb{P}(\sum_{i \in I} a_i = b) \leq c/\sqrt{n}$, and estimate c.

13 Tuesday, May 17, 2016

Midterm problem 1:

k-uniform hypergraph $\mathcal{H} = (V, \mathcal{E}), n = |V|, m = |\mathcal{E}|$.

Prove: $\tau(\mathcal{H}) \le \lceil \frac{n}{k} \ln m \rceil$ min cover (hitting set).

Pick sequence $x_1, \ldots, x_s \in V$ at random independently (with replacement).

$\mathbb{P}(x_i \in E) = k/n$,

$\mathbb{P}(x_i \notin E) = 1 - k/n$,

$\mathbb{P}(\forall i, x_i \notin E) = \mathbb{P}(\cap_{i=1}^{s} \text{"}x_i \notin E\text{"}) = (1 - k/n)^s$

$\mathbb{P}(\exists E : E \text{ is bad}) = \mathbb{P}(\cup_{E \in \mathcal{E}} E \text{ is bad}) \le m(1 - k/n)^2$

$1 - k/n < e^{-k/n}$.

$\therefore$ If $me^{-ks/n} \le 1$ then $\tau(\mathcal{H}) \le s$,

while $me^{-ks/n} \le 1 \iff \ln m - ks/n \le 0 \iff \ln m \le ks/n \iff s \ge \frac{n}{k} \ln m$

$\therefore$ If $s \ge \frac{n}{k} \ln m$, then $\tau \le s$.

With $s := \lceil \frac{n}{k} \ln m \rceil$ we have $\tau \le s$.

Midterm problem 2:

Def $\nu^*(\mathcal{H})$:

$g : \mathcal{E} \to \mathbb{R}$ is a fractional matching if $\forall E \in \mathcal{E}, g(E) \ge 0$ and $\forall v \in V, \sum_{E, v \in E} g(E) \le 1$.

$\text{value}(g) = \sum_{E \in \mathcal{E}} g(E)$

$\nu^*(\mathcal{H}) = \min_g \text{value}(g)$

If $\mathcal{H}$ is k-uniform then $\nu^* \le k\nu$.

$\mathcal{K}_n^{(k)}$: complete k-uniform hypergraph

$\nu = \lfloor n/k \rfloor, \nu^* = n/k$

$n/k = \binom{n}{k} / \binom{n-1}{k-1} \le \nu^* \le \tau^* \le n/k$ by uniform weight.

$\tau \le k\nu$

$\nu^* \le \tau^* \le \tau \le k\nu$ (first two parts trivial)

For infinitely many values of k, find $\mathcal{H}$ such that $\nu^*(k-1)\nu$

projective plane of order $k-1$: $n = (k-1)^2 + (k-1) + 1 = k^2 - k + 1$

$\nu = 1$. Need $\nu^* > k - 1$.

$\nu^* = \tau^* = n/k = (k^2 - k + 1)k = k - 1 + 1/k$ (uniform weight$= 1/k$)

"There is a difference between easy and trivial; trivial is straightforward."

Midterm problem 3:

$s \ge 2t + 1$ Kneser's graph

$K(s, t)$: $\binom{s}{t}$ vertices $\{v_T \mid T \subseteq [s], |T| = t\}$. $v_T \sim v_S$ if $T \cap S = \emptyset$

DO: $K(5, 2) =$ Petersen's, $n = \binom{5}{2} = 10$, $\deg = \binom{3}{2} = 3$.

Claim: $\alpha(K(s, t)) = \binom{s-1}{t-1}$

Lemma: A set, $A \subseteq V(K(s, t))$ is independent $\iff$ the corresponding labels are an intersecting hypergraph

the label of a set of vertices form a t-uniform hypergraph

max independence $\iff$ max intersecting form,
$\alpha = \binom{s-1}{t-1}$ EKR

Midterm problem 4:

$x_1 + \ldots + x_k = n$, $x_i \geq 2$, #solutions $= N(n,k)$
$y_1 + \ldots + y_k = n$, $y_i \geq 0$
$L(n,k) = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}$ (n stars)
Claim: $N(n,k) = L(n-2k, k)$
using $y_i := x_i - 2$

Midterm problem 5:

Flip n coins, $X =$ #pairs of consecutive heads
$E(X), \mathrm{Var}(X)$
$X = \sum_{i=1}^{n-1} Y_i$, where $Y_i = \begin{cases} 1 & \text{if } z_i = z_{i+1} = H \\ 0 & \text{if otherwise} \end{cases}$
$E(Y_i) = \mathbb{P}(z_i = z_{i+1} = H) = 1/4$
$E(X) = (n-1)/4$
$\mathrm{Var}(X) = \sum_i \sum_j \mathrm{Cov}(Y_i, Y_j) = \sum_{i=1}^{n-1} \mathrm{Var}(Y_i) + 2\sum_{i=1}^{n-2} \mathrm{Cov}(Y_i, Y_{i+1})$
$= 3(n-1)/16 + 2(n-2)/16$

Midterm problem 6:

$S(n,5) = \sum_{k=0}^{k} \binom{n}{5k}$
Claim: $\exists c > 0$ such that $|S(n,5) - 2^n/5| < (2-c)^n$
Note: $(2-c)^n/2^n = (1-c/2)^n \to 0$ exponentially
$S(n,5) = [(1+1)^n + (1+\omega)^n + (1+\omega^2)^n + (1+\omega^3)^n + (1+\omega^4)^n]/5$, $\omega = e^{2\pi i/5}$
$|S(n,5) - 2^n/5| = |\sum_{j=1}^{4}(1+\omega^j)^n|/5 \leq \frac{1}{5}\sum_{j=1}^{4} |(1+\omega^j)^n| \leq \frac{4(2-c)^n}{5} < (2-c)^n$
since $|1+\omega^j| < 1 + |\omega^j| = 2$, $\exists i > 0, \forall j, |1+\omega^j| < 2 - c$ (b/c ω^j not real for $j > 0$)

Ramsey's Theorem baby version: $n \to (k,l)$ if no matter how we color $E(K_n)$ red/blue, either $\exists$red K_k or $\exists$blue K_l (Erdös-Rado arrow notation)

Pf: of $6 \to (3,3)$

DO: (**Erdös-Szekeres**) $\binom{k+l}{k} \to (k+1, l+1)$. $k = l = 2 \implies \binom{4}{2} = 6 \to (3,3)$

Proceed by induction on $k+l$
Base cases $k=1$ or $l=1$
Inductive step:
Assume $k, l \geq 2$
Use inductive hypothesis both for $(k-1, l)$ and for $(k, l-1)$

HW: Prove $17 \to (3,3,3)$

Ramsey: $\forall k_1, \ldots, k_j, \exists n$ such that $n \to (k_1, \ldots, k_j)$

14 Thursday, May 19, 2016

Oddtown Theorem: $A_1, \ldots, A_m \subseteq [n]$, (i) $|A_i| =$ odd and (ii) $|A_i \cap A_j| =$ even $\implies m \leq n$.

Proof: incidence vectors $v_1, \ldots, v_m$– claim linear independence

DO: (a) If $v_1, \ldots, v_m \in \{0,1\}^n$ and $v_1, \ldots, v_m$ are linearly independent over $\mathbb{F}_p$, then they are linearly independent over $\mathbb{R}$. (b) Converse false for all p.

$$\text{incidence matrix } M = \begin{pmatrix} -v_1- \\ \vdots \\ -v_m- \end{pmatrix}$$

$MM^T = (|A_i \cap A_j|)_{m\times m}$
Claim MM^T has full rank (rk $= m$) over $\mathbb{F}_2$

DO: $\mathrm{rk}(AB) \leq \mathrm{rk}(A), \mathrm{rk}(B)$ over any field

Corollary: $m = \mathrm{rk}(MM^T) \leq \mathrm{rk}(M) = n$.

Observation with Erdös-Szekeres: $4^k > \binom{2k}{k} \to (k+1, k+1) =: (k+1)_2$. Moreover, $n = 4^k \to (k+1)_2$, and we can asymptotically estimate $n \to (1 + \frac{1}{2}\log_2 n)_2$.

"$n \to (k+l)$": in any graph G with n vertices either **clique number** $\omega(G) \geq k$ or $\alpha(G) \geq l$.

Want to find $n \not\to \binom{?}{k}_2$. Can try $n \not\to \binom{n}{2}_2$ or $n \not\to (\sqrt{n}+1)_2$.

[Look up Turán's Theorem.]

Erdös: $n \not\to (2\log_2 n)_2$

(We don't even know minimum number (Ramsey number) to arrow 5, i.e. ? $\to (5)_2$.)

Proof from Erdös: $n \not\to (k)_2$. Take a random graph $p = 1/2$ with space $|\Omega| = 2^{\binom{n}{2}}$ (uniform).
$A \subset [n]$, $|A| = k$,
$\mathbb{P}(A \text{ is a clique}) = 1/2^{\binom{n}{2}}$.
$\mathbb{P}(A \text{ is independent}) = 1/2^{\binom{n}{2}}$ likewise.
$\mathbb{P}(A \text{ homogeneous}) = 2/2^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$.
$\mathbb{P}(\exists \text{homogenous subset of size } k) < \binom{n}{k}2^{1-\binom{k}{2}}$ (inequality by union bound)
Corollary: If $\binom{n}{k}2^{1-\binom{k}{2}} \leq 1$, then $n \not\to (k)_2$.

(DO: $\binom{n}{k} \le \frac{n^k}{k!}$.)

if $\frac{2}{k!} \frac{n^k}{2^{\binom{k}{2}}} \le 1 \implies$

$\frac{2}{k!} \left(\frac{n}{2^{(k-1)/2}} \right)^k \le 1$?

sufficient to get $\frac{n}{2^{(k-1)/2}} \le 1$

for $n \not\to (k)_2$, it suffices: $\frac{n}{2^{(k-1)/2}} \le 1$

$n \le 2^{(k-1)/2}$

$\log_2 n \le \frac{k-1}{2}$

$k \ge 1 + 2 \log_2 n$.

Zsigmond Nagy's construction: gives $n \not\to (c\sqrt[3]{n})_2$

$n = \binom{s}{3}, V = \{v_T \mid T \subseteq [s], |T| = 3\}, T, S \subseteq [s], |T| = |S| = 3$. $v_T \sim v_S$ adjacent if $|S \cap T| = 1$.

HW: Prove Nagy's graph, $\omega(G) \le s$, $\alpha(G) \le s$. [For each part, use a theorem provided in class.]

$S \cap T \ne \emptyset$, so $\binom{s-1}{2} \sim s^2/2 \sim cn^{2/3}$

DO: Find $\Omega(s^2)$ triples in $[s]$ such that every pair intersects in 0 or 1.

Ramsey's Theorem: $\forall t, s, k_1, \ldots, k_s, \exists n$ such that $n \to (k_1, \ldots, k_s)^{(t)}$; $n \to (k)_s^{(t)}$ where t indicates coloring of t-tuples and $s = \#$colors.

Erdös-Rado: $n \to (c \log \log n)_2^{(3)}$

$n \to (\approx \frac{1}{2} \log_2 n)_2^{(2)}$ (graph case). So we find $r \approx \log_2 n$ vertices such that $\exists$sequence $c_1, \ldots, c_r$ of colors such that $\forall i < j, \text{color}(w_i, w_j) = c_i$. So, majority of the c_i is the same, say "blue" $W = \{w_i \mid c_i \text{ blue}\}$ so $|W| \gtrsim \frac{1}{2} \log_2 n$. Thus, W is all blue.

$2^{1+\ldots+r} = w^{(r+1)r/2} \approx 2^{r^2/2} \approx n$, so $r^2/2 \approx \log_2 n$, i.e., $r \approx \sqrt{2 \log_2 n}$. $\text{col}(w_i, w_j, w_l)$ only depends on (i, j). Pick $\frac{1}{2} \log_2 r$ out of these that are homogeneous $\sim \frac{1}{4} \log_2 \log_2 n$.

$t = 3 : m \not\to (c_1 \sqrt{\log n})_2^{(3)}$, but $n \to (c_2 \log \log n)$.

15 Tuesday, May 24, 2016

HW Problem reviewed:

Nagy's graph G_s: $n = \binom{s}{3}$#vertices, $\{v_T \mid T \subseteq [s], |T| = 3\}$, $v_{T_1} \sim v_{T_2}$ if $|T_1 \cap T_2| = 1$

Claim: $\alpha(G_s) \le s$ and $\omega(G_s) \le s$ (**clique number**)

Clique corresponds to triples $T_1, \ldots, T_m$ such that $\forall i \ne j, |T_i \cap T_j| = 1$

Need To Show: $m \leq s$, which we have by Fisher's Inequality

Independent set corresponds to $T_1, \ldots, T_m$ such that $\forall i \neq j, |T_i \cap T_j| = \begin{cases} 0 \\ 2 \end{cases}$

Need To Show: $m \leq s$, which we have by Oddtown Theorem

Recall the Ray-Chaudhuri – Wilson Inequality: If $A_1, \ldots, A_m$ uniform, $L = \{l_1, \ldots, l_s\}$, $A_i \subseteq [n]$, $\forall i \neq j |A_i \cap A_j| \in L$, then $m \leq \binom{n}{s}$

Also, recall from Frankl-Wilson (1980): If p prime, $A_1, \ldots, A_m \subseteq [n]$, $\forall i |A_i| = k$, $L = \{l_1, \ldots, l_s\}$, $k \notin L \mod p$, and $\forall i \neq j, |A_i \cap A_j| \in L \mod p$ (: $\forall i k \not\equiv l_i \mod p$), then $m \leq \binom{n}{s}$ ("**modular R-W theorem**")

Nagy: explicitly, $n \not\to (c\sqrt[3]{n})_2$

Explicit Ramsey numbers: $\forall \epsilon > 0, n \not\to (n^\epsilon)$, i.e. $n \not\to (n^{o(1)})$ with little-o notation.

Consider a FW graph, p prime. Then $n = \binom{2p^2-1}{p^2-1}$ subsets of $[2p^2-1]$ of size p^2-1: $\{v_T \mid T \subseteq [2p^2-1], |T| = p^2-1\}$, $v_{T_1} \sim v_{T_2}$ if $|T_1 \cap T_2| \equiv -1 \mod p$

Clique: $T_1, \ldots, T_m$ such that $\forall i \leq j, |T_i \cap T_j| \equiv -1 \mod p$

Claim: $m \leq \binom{2p^2-1}{p-1}$ (by R-W)

$|T_i \cap T_j| \in \{p-1, 2p-1, \ldots, p^2-p-1\}$ ($p-1$ elements, here)

Independence Number: $|T_i \cap T_j| \not\equiv -1 \mod p$, $|T_i| \equiv p^2 - 1 \equiv -1 \mod p$,

$L = \{0, 1, \ldots, p-2\} \implies \alpha = m \leq \binom{2p^2-1}{p-1}$

Note $\binom{2p^2-1}{p^2-1} \sim c \cdot 2^{2p^2-1}/\sqrt{2p^2-1} \sim c' \cdot 2^{2p^2}/p$, while $\binom{2p^2-1}{p} < (2p^2-1)^p = 2^{p\log_2(2p^2-1)}$

$p\log_2(2p^2-1) \sim p\log_2(p^2) = 2p\log_2 p$, and $\log(2^{2p^2}) = 2p^2$

$\implies \log(2^{2p^2})/\log[(2p^2-1)^p] \sim p/\log p$

$\implies \binom{2p^2-1}{p} \lesssim \binom{2p^2-1}{p^2-1}$, i.e. $n^{\log p/p} \lesssim n$ (consider $\log p/p = \epsilon$ here)

Projective plane $\mathcal{P} = (P, L, I)$, $I \subseteq P \times L$ (i.e. $p \multimap l$). Polarity is a bijection $f : P \to L$ such that $p_1 \multimap f(p_2)$ iff $p_2 \multimap f(p_1)$

DO: each Galois plane has a polarity ($p : [\alpha_1, \alpha_2, \alpha_3]$, $l : [\beta_1, \beta_2, \beta_3]$, $p \multimap l$ if $\sum \alpha_i \beta_i = 0$)

"Fixed point" of a polarity f: $p \multimap f(p)$; $[\alpha_1, \alpha_2, \alpha_3]$ is a fixed point of "standard" polarity if $\sum \alpha_i^2 = 0$. So $\mathbb{F}_q$ if q prime: $q | \alpha_1^2 + \alpha_2^2 + \alpha_3^2$.

Baer's Theorem: Every polarity has a fixed point, $\forall$ finite projective plane

For a contradiction, suppose f has no fixed point.

Consider an incidence matrix $M = (m_{ij})$; we list the lines in a given order, and list the points in the corresponding order from $l_i := f(p_i)$

If f is a polarity, then $M = M^T$ (symmetric matrix)

If f is fixed-point-free, then $\forall i, m_{ii} = 0$ (diagonal is all zero)
If M is an incidence matrix of a projective plane, then

$$M^T M = \begin{pmatrix} n+1 & 1 & \cdots & 1 \\ 1 & n+1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & n+1 \end{pmatrix} = J + nI$$

row sum $= (n+1) + N - 1 = n^2 + 2n + 1 = (n+1)^2$
all others: for $i \geq 2, Je_i = 0$ so $M^2 e_i = \lambda_i^2 e_i$, so $\lambda_2^2 = \ldots = \lambda_N^2 = n$ ($\lambda_i = \pm\sqrt{n}$),
so now $M^2 = J + nI$
Spectral Theorem: M has an orthonormal eigenbasis $e_1, \ldots, e_N$:

$$Me_i = \lambda_i e_i, Me_1 = (n+1)e_1, \; e_i \perp e_j, \text{ and } e_1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

So, $\text{trace}(M) = \sum_i m_{ii} = 0 = \sum_i \lambda_i = (n+1)^2 + \sqrt{n} \pm \ldots \sqrt{n}$ ($n^2 + n$ terms of form $\pm\sqrt{n}$)
Thus, $0 = (n+1) + K\sqrt{n} \implies 0 \mod n \equiv -K\sqrt{n} = (n+1) \implies K^2 n \equiv (n+1)^2 = 1 \mod n$

16 Thursday, May 26, 2016

Quiz 3 Problem 1:

$x_1 + \ldots + x_k = n$ count solutions in positive odd integers
$y_i := (x_i - 1)/2$ integer ≥ 0, so $\sum y_i = (n-k)/2, y_i \geq 0$ (bijection b/w sets of solutions)
Case 1: $n - k$ odd $\implies$ #solutions $= 0$
Case 2: $n - k$ even $\implies \binom{\frac{n-k}{2}+k-1}{k-1} = \binom{\frac{n+k}{2}-1}{k-1}$

Quiz 3 Problem 2:

$\alpha(G) \leq 1 + 2\log_2 n$ for almost all grpahs G with n vertices
$p_n = \mathbb{P}$(for random graph G with n vertices, this holds)
$\lim_{n\to\infty} p_n = 1$
In class: $1 - p_n = \mathbb{P}(\exists$ independent set of size $> 1 + 2\log_2 n) \leq 1/k!, k : 1 + 2\log_2 n$

Quiz 3 Bonus:

Prove: $\forall k \exists n$ from any n points in the plane, no 3 on a line, $\exists k$: convex k-gon

DO: k points span a convex k-gon $\iff$ every 4 of them span a convex 4-gon (quadrilateral)

Color quadruples of points: red if convex, blue if concave.
$n \to (k, 5)_2^{(4)}$; 5-points all-blue impossible; therefore k-point all-red set exists.

Comment: this gives astronomically large bound, $n = 2^{2^{2^k}}$

"**Friendship graph**:" every pair of points has exactly 1 common neighbor

Example: "bouquet of triangles:" a set of triangles that share one point and are disjoint otherwise.

Theorem (Erdős, Rényi, Vera Sós): Bouquets of triangles are the only Friendship graphs.

Pf: $N(v)$: set of neighbors of v
$\forall v, w, |N(v) \cap N(w)| = 1$
$\forall x, y \exists! v$ such that $x, y \in N(v)$
$\therefore \{N(v) \mid v \in V\}$ is a possibly degenerate projective plane
Case 1: degenerate. Then there is a vertex adjacent to all the others.
Exercise: this must be a bouquet of triangles
Case 2: Projective plane. Claim: cannot happen
$v \leftrightarrow N(v)$; polarity: $x \in N(y) \iff y \in N(x)$ because
in Friendship graph, $x \in N(y) \iff x \sim y$ (adjacent) $\iff y \in N(v)$

Baer's Theorem: Every polarity of a finite projective plane has a fixed point: $v \in N(v)$.

But this is impossible in our case because it would mean $v \sim v$. So this case cannot arise from a Friendship graph.

Linear Programming Problem:

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, (Notation: $\underline{x} \leq \underline{y}$ if $\forall i, x_i \leq y_i$)). Concise notation for a system of k linear equations in n unknowns: $A\underline{x} = \underline{b}, A = (a_{ij})_{k \times n}, \underline{x} \in \mathbb{R}^n, \underline{b} \in \mathbb{R}^k$.

Here, we instead have constraints $A\underline{x} \leq \underline{b}, \underline{x} \geq \underline{0}$ with the objective $\max \leftarrow \underline{c}^T \cdot \underline{x} = \sum_{i=1}^n c_i x_i$ ($\underline{c} \in \mathbb{R}^n$) (**Primal Linear Program**). We call a set of constraints **feasible** if $\exists$ solution. The **Dual Linear Program** has constraints $A^T \underline{y} \geq \underline{c}, \underline{y} \geq \underline{0}$ and objective $\min \leftarrow \underline{b}^T \cdot \underline{y} = \sum_{j=1}^k b_j y_j$.

Proposition (mini-theorem): $\forall \underline{x}, \underline{y}$, if $\underline{x}$ satisfies the Primal Linear Program and $\underline{y}$ satisfies the Dual Linear Program, then $\underline{c}^T \underline{x} \leq \underline{b}^T \underline{y}$

$\therefore \max(\text{primal}) \leq \min(\text{dual})$

Proof: $\underline{c} \leq A^T \underline{y} \iff c^T \leq y^t A$
$\underline{c}^T \cdot \underline{x} \leq \underline{y}^T A \underline{x} \leq y^T b$

The two sides of this inequality are in fact equal.

Duality Theorem of Linear Programming: If both the primal and the dual are feasible, then max(primal) = min(dual)

DO!!: Infer $\nu^* = \tau^*$ from LP duality.

Lovász: (a) $\tau^* \leq \tau \leq \tau^*(1+\ln \deg_{\max})$, where $\deg_{\max}$ is the max degree in the hypergraph ("**integrality gap**"). (b) Greedy algorithm finds such a cover

Permanent Inequality: $A \in M_n(\mathbb{R}), \operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i,\sigma(i)}$; stochastic matrix $A = (a_{i,j}) \in M_n(\mathbb{R}), a_{i,j} \geq 0, \forall i \sum_j a_{i,j} = 1$; doubly stochastic A has A^T also stochastic, i.e., columns each sum to 1.

Exercise: If A stochastic, then $\operatorname{per}(A) \leq 1$.

Pf: $1 = \prod_{i=1}^n (\sum_{j=1}^n a_{ij}) = \sum(n^n \text{ terms }) \geq \sum(n! \text{ terms }) = \operatorname{per}(A)$

Note J : all ones has $\frac{1}{n}J$ doubly stochastic and $\operatorname{per}(\frac{1}{n}J) = n!/n^n$, while $\operatorname{per}(I) = 1$ and I is doubly stochastic matrix.

DO: Assume A is stochastic. Prove: $\operatorname{per}(A) = 1 \iff A$ is a permutation matrix

Permanent Inequality: If A is doubly stochastic, then $\operatorname{per}(A) \geq n!/n^n$

Recall $n!/n^n > e^{-n}$. (First, note $e^x = \sum_{k=0}^\infty x^k/k! > x^k/k!$, so $e^n > n^n/n! \iff n!/n^n > e^{-n}$.) We can use this to find the asymptotic log of the number of Latin squares of order n, $L(n) := \#\{n \times n \text{ Latin squares}\}$.

Theorem: $\ln L(n) \sim n^2 \ln n$

i.e., $L(n) < n^{n^2}$. Need to find lower bound on $L(n)$.

#perfect matchings in a bipartite graph (n, n): $\operatorname{per}(A)$ for the incidence matrix A

r-regular: $\frac{1}{r}A$ doubly stochastic, so $\frac{1}{r^n}\operatorname{per}(A) = \operatorname{per}(\frac{1}{r}A) > e^{-n} \implies \operatorname{per}(A) > (\frac{r}{e})^n$

$L(n) > \prod_{r=1}^{n-1} (\frac{r}{e})^n > \frac{[(n-1)!]^n}{e^{n^2}}$

$\ln L(n) > n \ln[(n-1)!] - n^2 \sim n^2 \ln n$