CMSC-27410/37200 Honors Combinatorics
First Midterm May 12, 2016

Instructor: László Babai Ryerson 164 e-mail: `laci@cs`

This exam contributes 14% to your course grade. Take this problem sheet home and solve it again on your time.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with the ordinary problems.

1. (24 points) Let $\mathcal{H} = (V, \mathcal{E})$ be a k-uniform hypergraph $|V| = n$ vertices and $|\mathcal{E}| = m$ edges. Prove:
$$\tau(\mathcal{H}) \le \left\lceil \frac{n}{k} \ln m \right\rceil .$$
Here $\tau(\mathcal{H})$ is the covering number of $\mathcal{H}$, i.e., the size of the smallest subset of V that intersects every edge.

2. (6+8+16 points) (a) Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Define fractional matchings and the fractional matching number $\nu^*(\mathcal{H})$. (b) If $\mathcal{H}$ is k-uniform, prove that $\nu^* \le k\nu$. (c) For infinitely many values of k, find a k-uniform hypergraph such that $\nu^* > (k-1)\nu$. State and prove the values of ν and ν^* for your hypergraphs. — For (b) and (c), you may use results stated in class even if we did not prove them in class; state the results you use. (This comment applies to this problem only.)

3. (18 points) Kneser's graph $K(s,t)$ is defined for integers s, t where $t \ge 2$ and $s \ge 2t+1$ as follows. There are $n = \binom{s}{t}$ vertices labeled as v_T where $T \in \binom{[s]}{t}$ (so there are $n = \binom{s}{t}$ vertices; each vertex corresponds to a t-subset of $[s]$); and two vertices v_A and v_B are adjacent if the corresponding t-sets A, B are disjoint. Prove: $\alpha(K(s,t)) = \binom{s-1}{t-1}$. Here α denotes the independence number. You may use a theorem proved in class; name the theorem.

4. (16 points) Find the number of solutions of the equation $x_1 + \cdots + x_k = n$ in integers $x_i \ge 2$. Your answer should be a simple closed-form expression in terms of n and k. Prove your answer.

5. (6+22 points) We flip n fair coins and obtain a random sequence of Heads and Tails, e.g., $HTTHHHT$. Let X denote the number of consecutive pairs of Heads. E.g., $X(HHTTHHHT) = 3$ and $X(HHHHH) = 4$. Compute (a) $E(X)$ and (b) $\mathrm{Var}(X)$. Give your answers as polynomials of n in standard form $(a_0 + a_1 n + \dots)$.

6. (24 points) Let $S(n,5) = \sum_{k=0}^{\infty} \binom{n}{5k}$. Prove: there exists a constant $c > 0$ such that for all sufficiently large n we have

$$\left| S(n,5) - \frac{2^n}{5} \right| < (2-c)^n.$$

7. (BONUS 15 points) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, \dots$. Suppose $f(x)$ has the closed-form expression $f(x) = \frac{g(x)}{(1-x)^k}$ where g is a polynomial. Assume $g(1) \neq 0$. Prove: $a_n \sim cn^k$ for some constant $c \neq 0$. (In this asymptotic equality, k is fixed and $n \to \infty$.) Express c as a function of k and the polynomial g. Your expression should be very simple, fewer than ten symbols.

8. (BONUS 5+8+20 points) Define $\chi^*(G)$, the fractional chromatic number of the graph G, as follows: Let V be the set of vertices of G and let $C_1, \dots, C_m$ denote the independent sets in G. Let $x_1, \dots, x_m$ be real variables. For every vertex $v \in V$ consider the quantity $s(v)$ which is the sum of those x_j for which $v \in C_j$. A *fractional coloring* assigns values $x_j \geq 0$ such that $s(v) \geq 1$ for each vertex v. The value of a fractional coloring is $\sum_{j=1}^{m} x_j$. Let $\chi^*(G)$ be the minimum value of a fractional coloring.

   (a) Prove: $\chi^*(G) \leq \chi(G)$.

   (b) Compute $\chi^*(C_5)$ where C_5 is the cycle of length 5. Prove your answer.

   (c) Prove: $\alpha(G)\chi^*(G) \geq n$.