Honors Combinatorics CMSC-27410/Math-28400/CMSC-37200 First Quiz. April 5, 2016
Instructor: László Babai

Name (print): ______________________ **Major/Year** ______________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may **continue on the reverse.** This quiz contributes 6% to your course grade.

Warning: the BONUS problems are underrated; solve them only after you solved the ordinary problems. You are not expected to solve the bonus problems unless you seek graduate credit for this class.

Note: after the test, the point values have been slightly reduced so the points for the non-bonus problems add up to 60, corresponding to the 6% contributions of this test to the grade.

1. (10 points) (orig. 12 pts) Given a set V of n elements, count the simple k-uniform hypergraphs with vertex set V. ("Simple" means there are no multiple edges.) Give a simple closed-form expression (no summation signs or dot-dot-dots). Do not prove.

2. (17 points) (orig. 18 pts) Recall: the incidence matrix of a hypergraph with n vertices and m edges is an $m \times n$ $(0, 1)$-matrix (each entry is 0 or 1); the entry in position (i, j) is 1 if vertex number j belongs to edge number i. Let M be the incidence matrix of a projective plane of order n. Let $N = n^2 + n + 1$; so M is an $N \times N$ matrix. Compute the matrix $M^T M$. (M^T is the transpose of M, so the (i, j) entry of M^T is the (j, i)-entry of M.) Give a very simple description of the (i, j) entry of $M^T M$. Briefly reason your answer.

3. (17 points) (orig 18 pts) Prove that there are at most 12 pairwise orthogonal Latin squares of order 14. Use results stated in class; state the results you use. Based on those results, your proof should be just a couple of lines.

4. (16 points) (orig. 18 pts) Let V be a set of n elements, $n \geq 3$. Find 2^{n-1} subsets of V such that they intersect pairwise but all of them do not intersect (their intersection is empty). Your description should be very simple.

5. (BONUS)

   (a) (7 points) (orig. 8 pts) Prove: a maximal intersecting simple hypergraph is maximum. ("Maximal": no edge can be added; "maximum": it has maximum possible number of edges, i.e., in this case 2^{n-1} edges.)

   (b) (7 points) (orig. 8 pts) Prove that part (a) fails for uniform hypergraphs: for every $k \geq 2$ and $n \geq 2k+1$ construct a maximal k-uniform simple hypergraph that is not maximum. (Recall that the maximum in this case is $\binom{n-1}{k-1}$ by the Erdős–Ko–Rado Theorem.)