Honors Combinatorics CMSC-27410/Math-28400/CMSC-37200
Second Quiz. April 26, 2016
Instructor: László Babai

Name (print): ________________________________ **Major/Year** ____________________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may **continue on the reverse.** This quiz contributes 5% to your course grade.

Warning: the BONUS problems are underrated; solve them only after you solved the ordinary problems. You are not expected to solve the bonus problems unless you seek graduate credit for this class.

1. (6+8 points) Consider the sequence $a_n = \binom{2n}{n}$.

   (a) Determine the largest value r such that $a_n x^n \to 0$ for all x with $0 \leq x < r$.

   (b) Find a closed-form expression of the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$. (Hint. The number r from part (a) is the convergence radius of $f(x)$.)

   Reason your answers.

2. (12 points) The drunkard has $\binom{2n}{n}$ ways of walking $2n$ steps, starting and endig at zero. How many of these $\binom{2n}{n}$ walks will ever reach point k (k is a positive integer)? Your answer should be a very simple closed-form expression. Give a very simple proof of your answer; name the principle used.

3. (6 points) For all $k, \ell \geq 1$ find a sequence of $k\ell$ numbers such that the sequence contains no increasing subsequence of length $k+1$ and no non-increasing subsequence of length $\ell + 1$.

4. (3+4+6+5 points) We roll n dice; the numbers shown are $X_1, \ldots, X_n$ $(1 \leq X_i \leq 6)$.
(a) What is the size of the sample space for this experiment? — Let S be the sum and P the product of the X_i. Determine (b) $E(S)$ (c) $\mathrm{Var}(S)$ (d) $E(P)$. Your answer to each question should be a very simple formula. Give a brief reason for your answers, no detailed proofs required.

5. (BONUS 4 points) A *homogeneous third-order recurrence* is a recurrence of the form $b_n = \alpha b_{n-1} + \beta b_{n-2} + \gamma b_{n-3}$ for some constants α, β, γ. Find a homogeneous third-order recurrence satisfied by the sequence $b_n = n^2$.

6. (BONUS 12 points) Let $p(n)$ denote the integer partition number (number of solutions to the equation $n = \sum_{i=1}^{n} ik_i$ in nonnegative integers k_i). Outline the proof that $\ln p(n) < \pi\sqrt{2n/3}$. Points are awarded depending on specificity and accuracy of the details.