CMSC-37110 Discrete Mathematics
FINAL EXAM 10:30 – 12:30 Thursday, December 8, 2016

Name (print): ________________________________

This exam contributes 35% to your course grade.

Do **not** *use book, notes, scrap paper.* Use the space provided; you may continue on the **back of each sheet. Show all your work.** If you are not sure of the meaning of a problem, **ask the proctor.** The XC problems are **underrated**, work on them after having done your best on the non-XC problems.

READ THIS!

In all problems involving a graph G, we use the following notation: n denotes the number of vertices, m the number of edges, $\chi(G)$ the chromatic number.

If a problem speaks about a random element of a given set and does not specify a distribution, the distribution is assumed to be uniform.

You may use results proved in class except if the question is essentially the same as the result proved in class; in the latter case you are asked to reproduce the proof.

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1. (2 points) Assume on average the instructor asked 18 questions in each class. Write your answers in the format "SIX (6)"

   (a) On average, how many times did you raise your hand to answer a question in each class? Give a rough estimate.

   (b) On average, how many times did you **not** raise your hand to answer a question in each class, even though you felt you had an answer?

2. (8+10 points)

(a) Let a_n be a sequence of real numbers and L a real number. Define the statement "$\lim_{n\to\infty} a_n = L$." Your answer should be a well-quantified expression with no English words.

(b) For a real number L, consider the following statement $S(L)$:

($S(L)$) If a_n and b_n are sequences of real numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$ then $a_n \sim b_n$.

Decide, for what values of L is statement $S(L)$ true. Prove your answer.

3. (7 points) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Recall that for $k \geq 1$ we have $A^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$ where F_k is the k-th Fibonacci number. Use this fact to give a very simple proof of the following identity: $F_{k+1}F_{k-1} - F_k^2 = (-1)^k$. (One line, no calculation.) Hint: determinants.

4. (7+8+8 points)

(a) Let $A \in M_n(\mathbb{R})$ be an $n \times n$ real matrix. Define the concept of an eigenbasis of A. Make sure to clearly state, a basis of what space we are talking about.

(b) Assume the characteristic polynomial of a 4×4 real matrix B is $f_B(t) = (t^2 - 3)(t^2 - 4t + 3)$. Prove: B has an eigenbasis.

(c) We say that two $n \times n$ matrices are *isospectral* if they have the same characteristic polynomial.
Find two isospectral 2×2 real matrices A and B such that A has an eigenbasis and B does not have an eigenbasis. Prove your answers.

5. (14 points) Let $A \in M_n(\mathbb{R})$ be a symmetric real matrix. Let λ be the largest eigenvalue of A. Prove: for every $v \in \mathbb{R}^n$ we have $v^T A v \le \lambda \|v\|^2$. Hint. Use the Spectral Theorem. Express v as a linear combination of an orthonormal eigenbasis.

6. (15 points) Let a, b be relatively prime positive integers. Prove: all sufficiently large positive integers n can be written as $n = ax + by$ where x, y are *positive* integers. Estimate, in terms of a and b, how large is "suffiiently large" in this statement.

7. **(9+9+9XC points)** The *average degree* of a graph $G = (V, E)$ is defined as

$$D = \frac{1}{n} \sum_{v \in V} \deg(v).$$

(a) Assume $m \geq 1$. Prove: $D < \sqrt{2m}$.

(b) Assume $m \geq 1$. Prove: $\chi(G) \leq \lceil \sqrt{2m} \rceil$.

(c) **(XC)** Let λ denote the largest eigenvalue of the adjacency matrix of G. Prove: $\lambda \geq D$.

8. **(14 points)** Let p, q, r be distinct odd primes. Determine the number of solutions modulo pqr of the congruence $x^2 \equiv 1 \pmod{pqr}$.

9. (4+12+5+12XC points)

(a) Define stochastic matrices.

(b) Let $A \in M_n(\mathbb{R})$ be a stochastic matrix and $\lambda \in \mathbb{C}$ a complex eigenvalue of A. Prove: $|\lambda| \leq 1$. Do not use the Perron–Frobenius Theorem.

(c) Give an example of an $n \times n$ stochastic matrix in which the eigenvalue 1 has algebraic multiplicity n. (Solve this problem for every n.)

(d) (XC) Count the solutions to part (c). Prove your answer.

10. (5+6+5+10XC+8 points)

(a) Given a finite probability space (Ω, P), define the concept of a random variable over (Ω, P).

(b) Let X, Y be random variables over (Ω, P). Define what it means that X, Y are independent.

(c) Define what it means that X, Y are uncorrelated.

(d) (XC) Prove: If X, Y are independent then they are uncorrelated.

(e) Give an example of two random variables that are uncorrelated but not independent. Prove. Make your sample space as small as possible. You don't need to prove minimality.

11. (3+5+5+6+8+7 points)

(a) Define what it means that R is a relation on the set A.

(b) If $|A| = n$, what is the number of relations on A ?

(c) What is the number of symmetric relations on A ?

(d) Let $B(n)$ denote the number of equivalence relations on A. Prove:

(I) $B(n) \leq n^n$

(II) For all k satifying $1 \leq k \leq n$ we have $B(n) \geq k^{n-k}$.

(III) $\ln B(n) \sim n \ln n$

12. (14 points) Recall that a DAG (directed acyclic graph) is a digraph with no directed cycles. A *topological sort* of a digraph is a linear ordering (numbering) of its vertices, $v_1, \dots, v_n$, such that if $v_i \to v_j$ is an edge then $i < j$. Prove: every DAG has a topological sort.

13. (5+8 points)

    (a) State the Prime Number Theorem (PNT). Define the notation you use.

    (b) We pick a 100-digit number at random; initial zeros are permitted. Estimate the probability that the number picked is prime. Use the PNT; assume the concrete number in question is "large enough." Your answer should be of the form $1/N$ where N is an integer. Determine N. Do not use a calculator. Use the value $\ln 10 \approx 2.3026$.

14. (3+5+7+15 points) Recall that an *orientation* of a graph G is a digraph obtained by assigning a direction to each edge of G. So the number of orientations of a graph is 2^m.

(a) Let N denote the number of 3-cycles in the complete graph K_n. Determine N.

(b) Let T_n denote a random orientation of K_n. (So T_n is a random tournament.) What is the size of the sample space for this experiment?

(c) Let X denote the number of directed 3-cycles in T_n. Determine $E(X)$.

(d) Determine $\mathrm{Var}(X)$.

Your answers should be simple closed-form expressions. Prove your answers.

15. (7XC points) Let H_n denote the $n \times n$ grid graph; let u be its bottom left corner and v its top right corner. Consider a random orientation of $\vec{R}_n$ of H_n. Prove that in $\vec{R}_n$, almost surely v is not accessible from u by a shortest grid-path (of length $2n-2$). [Note: the condition of limiting the search to the shortest grid-paths was erroneously omitted in the test. Because of this error, the point value of the problem was reduced from 14 points to 7 points.]

16. (5+5+5+8 points)

(a) Define a stationary distribution of a finite Markov Chain (MC).

(b) Define ergodic finite Markov Chains. Define the concepts you use in the definition.

(c) Draw a MC that is irreducible (its associated digraph is strongly connected) but not ergodic. Use as few states as possible. Make sure to indicate all possible transitions. State why your MC is not ergodic.

(d) Find an ergodic finite MC such that its (unique) stationary distribution is not uniform. Prove. Use as few states as possible. (You don't need to prove minimality.)

17. (5+10XC points)

(a) Define the diameter of a graph.

(b) (XC) Prove: almost all graphs have diameter 2.

18. (5+15 points)

(a) Define what it means that a digraph is strongly connected. Make your definition as simple as possible, using only the notion of directed paths. (Do not refer to "strong components.")

(b) Recall that a *cut* of a digraph $G = (V, E)$ is a partition (A, B) of V into $V = A \dot\cup B$ (disjoint union) where A, B are not empty and $E(B, A) = \emptyset$ where $E(B, A)$ denotes the set of edges from B to A. Prove: if G is not strongly connected then G has a cut.

19. (8XC points) Let $A \in M_n(\mathbb{Z})$ with $\det(A) = 1$. Let $x = (x_1, \ldots, x_n)^T \in \mathbb{Z}^n$ and let $y = (y_1, \ldots, y_n) = Ax$. Prove: $\gcd(x_1, \ldots, x_n) = \gcd(y_1, \ldots, y_n)$.

20. (10+8XC points) Let X be the length of the longest increasing sequence in a random permutation of the numbers $\{1, 2, \ldots, n\}$.

   (a) Prove: $E(X) = \Omega(\sqrt{n})$.

   (b) (XC) Prove: $E(X) = O(\sqrt{n})$.

21. (5+5+5+5+5 points) A matrix $A \in M_n(\mathbb{R})$ is nonsingular if and only if (complete each statement; each statement should be a necessary and sufficient condition)

   (a) $\det(A)$

   (b) $\operatorname{rk}(A)$

   (c) the inverse of A

   (d) the system of equations $Ax = 0$

   (e) eigenvalue

22. (6+8+6+8XC points)

(a) Define odd and even permutations. Define the concepts you use in your definition.

(b) For $n \geq 2$, consider the permutation that permutes n elements in a cycle: $a_1 \to a_2 \to \dots a_n \to a_1$ (the n-cycle). Is this permutation even or odd? (The answer depends on n.) Prove your answer.

(c) Prove: for $n \geq 2$, exactly half of the $n!$ permutations of n objects is even.

(d) (XC) Prove: exactly half of the 16! starting positions of Sam Lloyd's 15-puzzle are solvable.

23. (4XC+4XC points)

(a) Let A be a nonsingular $n \times n$ matrix. Prove: one can change an entry in the first row of A to turn it into a singular matrix.

(b) Find a nonsingular 3×3 matrix $B = (b_{ij})$ such that no matter how we change b_{11} (the top left entry), the matrix remains nonsingular.