Discrete Math 37110 - Class 6 (2016-10-13)

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6.1 Asymptotic notation

Please review the instructor's online "Discrete Mathematics" lecture notes (LN) for asymptotic notation. Here are the definitions.

Notation 6.1. 1. $a_n \sim b_n$ (read: a_n is asymptotically equal to b_n) if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. (Note: this requires that b_n be **almost never zero**, i.e., $b_n = 0$ happens only for a finite number of values of n.)

2. "Little-oh" notation: $a_n = o(b_n)$ (read: "a_n is little-oh of b_n") if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Example 6.2.

$$\frac{5n^6 + \sqrt{n}}{20n^4 + 1} \sim \frac{n^2}{4}$$

Proof. Let us denote the left-hand side by a_n, so $a_n = \frac{5n^6+\sqrt{n}}{20n^4+1}$. Notice that the left-hand side can be written as $a_n = n^2 c_n$ where

$$c_n = \frac{5 + \frac{\sqrt{n}}{n^6}}{20 + \frac{1}{n^4}}.$$

The numerator of this expression approaches 5, the denominator appraches 20, so $\lim_{n\to\infty} c_n = 5/20 = 1/4$. So $\frac{a_n}{n^2} \to \frac{1}{4}$ and therefore $\frac{4a_n}{n^2} \to 1$. In other words, $\frac{a_n}{n^2/4} \to 1$, which means $a_n \sim n^2/4$. □

Remark 6.3. The notation $c_n \to 1/4$ has the same meaning as $\lim_{n\to\infty} c_n = 1/4$. But you cannot write $a_n \to n^2/4$ because $\lim_{n\to\infty} a_n = \infty$, or, in other notation, $a_n \to \infty$.

You can write $c_n \sim 1/4$ where the right-hand side is interpreted as the constant sequence $1/4, 1/4, 1/4, \ldots$, whereas when writing $c_n \to 1/4$, the "1/4" refers to a number, not a sequence.

Remark 6.4. While we write $7n^5 = o(n^6)$, we say "$7n^5$ **is** $o(n^6)$." We do not say "equal" for this equality sign; in fact, it is not symmetric; read it from left to right.

Notation 6.5 (big-Oh)**.** We say that $a_n = O(b_n)$ (read: "a_n is big-Oh of b_n") if $(\exists C)(\exists N)(\forall n)(n \geq N \implies |a_n| \leq C\,|b_n|)$. In other words, $(\exists C)$ such that the inequality $|a_n| \leq C\,|b_n|)$ holds **for all sufficiently large** n.

Remark 6.6. If $a_n = O(b_n)$, we say that the **rate of growth** of a_n is less than or equal to the rate of growth of b_n. C is referred to as the **implied constant**.

Example 6.7. $1000n^6 + 100n^2 = O(n^6)$. The implied constant is any $C > 1000$ (e.g., $C = 1001$).

Notation 6.8 (big-Omega)**.** We say that $a_n = \Omega(b_n)$ (read: "a_n is big-Omega of b_n") if $b_n = O(a_n)$.

Remark 6.9. Once again, note that $=$ is not an equality. We read it as "is." O and Ω are reflexive and transitive relations on the set of sequences, but it is not a symmetric relation.

DO 6.10. Show that $\sim$ is an equivalence relation on the set of *almost-never-zero* sequences (sequences that have only a finite number of terms equal to zero).

DO 6.11. Show o is reflexive and transitive on the set of almost-never-zero sequences.

Notation 6.12 (big-Theta)**.** We say that $a_n = \Theta(b_n)$ (read: "a_n is big-Theta of b_n") if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$. If this is the case, we say that a_n and b_n have the **same rate of growth**.

DO 6.13. Prove: $a_n = \Theta(b_n)$ if and only if there exist **positive** constants C_1, C_2 such that $C_1|b_n| \leq |a_n| \leq C_2|b_n|$ holds for all sufficiently large n.

DO 6.14. Prove: if $a_n \sim b_n$ then $a_n = \Theta(b_n)$.

DO 6.15. Show that Θ is an equivalence relation on sequences.

DO 6.16. Assume $a_n = O(b_n)$ and $a_n = O(c_n)$. (a) Prove: the relation $a_n = O(b_n + c_n)$ does not follow from these assumptions. (b) Prove: $a_n = O(b_n + c_n)$ doesfollow if $a_n, b_n \geq 0$.

DO 6.17. If $a_n = O(c_n)$ and $b_n = O(c_n)$ then $a_n + b_n = O(c_n)$.

Example 6.18. The notation $a_n = o(1)$ means precisely that $a_n \to 0$.

The notation $b_n = O(1)$ means precisely that b_n is bounded, i.e., $(\exists C)$ such that $|b_n| \leq C$ holds for all sufficiently large n.

Definition 6.19. If $c_n = \Omega(1)$, we say that c_n is **bounded away from** 0, meaning that $(\exists c > 0)$ such that $|c_n| > c$ for all sufficiently large n,

Remark 6.20 (Origin of the asymptotic notation)**.** The $\sim$, little-oh and big-Oh notation ("Landau notation") originates from number theory around 1900. Donald Knuth introduced Θ, Ω, and ω for computer science in the early 1970s. ($a_n = \omega(b_n)$ means $b_n = o(a_n)$; we shall not use this notation.)

Remark 6.21. Asymptotic relations like $a_n \sim n^2/4$ or $a_n = O(n^2)$ are insensitive to changing a finite number of terms in the sequence a_n; we even permit a finite number of terms to be undefined. In particular, such relations reveal nothing about a particular term in the sequence, such as a_{1000}.

6.2 Graph Theory

DO 6.22. Study **graph terminology** from the instructor's online "Discrete Mathematics" lecture notes (LN).

Definition 6.23. Graphs have nodes, called "vertices," and links, called "edges." The singular of the word "vertices" is vertex. There are no such words as "vertice" or "vertexes" or "verticies."

The example in class had 6 vertices and 7 edges. We write $G = (V, E)$ to denote a graph with vertex set V and edge set E. The egdes are unordered pairs of distinct vertices. The example in class was the graph (V, E) with $V = [6]$ and
$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}, \{6, 2\}\}$.

Notation 6.24. If A is a set, then $\binom{A}{k}$ denotes the set of k-subsets of A.

Remark 6.25. With this notation, $E \subseteq \binom{V}{2}$.

Definition 6.26 (Adjacency)**.** Two vertices $v, w \in V$ are **adjacent** in G (notation: $v \sim_G w$) if $\{v, w\} \in E$. In this case we also say that v and w are **neighbors**. We also simply write $v \sim w$ if the graph G is clear from the context. The **adjacency relation** is an irreflexive and symmetric relation on the set V.

In this class, most of the time we use the notation $n = |V|$ and $m = |E|$, so our graphs will have n vertices and m edges, unless expressly stated otherwise.

DO 6.27. $0 \le m \le \binom{n}{2}$.

DO 6.28. (a) The number of graphs on a given set V of n vertices is $2^{\binom{n}{2}}$.
(b) The number of graphs on a given set V of n vertices that have m edges is

$$\binom{\binom{n}{2}}{m}.$$

Definition 6.29. The **complete graph (clique)** K_n has all pairs adjacent (and thus it has $m = \binom{n}{2}$ edges). So a graph (V, E) is complete exactly if $E = \binom{V}{2}$.

Definition 6.30 (Subgraph)**.** We say that $G = (V, E)$ is a **subgraph** of $H = (W, F)$ (notation: $H \subseteq G$) if $W \subseteq V$ and $F \subseteq E$.

Notation 6.31 (Cycles, paths)**.** We denote the cycle of length n by C_n ($n \ge 3$). (See LN for the definition.) C_n has n vertices and n edges.

We denote the path of length $n - 1$ by P_n ($n \ge 1$). P_n has n vertices and $n - 1$ edges.

Definition 6.32 (Induced subgraph)**.** Given the graph $G = (V, E)$ and $W \subseteq V$, the subgraph of G **induced on the subset** W is the graph $G[W] = (W, E \cap \binom{W}{2})$.

DO 6.33. The number of induced subgraphs of G is 2^n.

Hint. You only need to specify the vertices, not the edges of an induced subgraph.

Definition 6.34 (Spanning subgraph)**.** Given the graph $G = (V, E)$ and $F \subseteq E$, we call the graph $H = (V, F)$ a **spanning subgraph** of G.

DO 6.35. The number of spanning subgraphs of G is 2^m.

Hint. You only need to specify the edges, not the vertices of a spanning subgraph.

Definition 6.36 (Complement)**.** The **complement** of a graph $G = (V, E)$ is $\overline{G} = (V, \overline{E})$ where $\overline{E} = \binom{V}{2} \setminus E$. So for any $v \neq w \in V$ we have $v \sim_{\overline{G}} w \iff v \not\sim_G w$.

Definition 6.37 (Isomorphism)**.** Given graphs $G = (V, E)$ and $H = (W, F)$, a function $f : V \to W$ is a $G \to H$ **isomorphism** if f is a bijection that preserves adjacency, i.e., $(\forall u, v \in V)(u \sim_G v \iff f(u) \sim_H f(v))$.

Definition 6.38 (Isomorphic graphs)**.** We say that the graphs G and H are **isomorphic** (notation: $G \cong H$) if there exists a $G \to H$ isomorphism.

DO 6.39. Isomorphism is an equivalence relation on graphs.

Definition 6.40 (Degree)**.** The degree $\deg(v)$ of the vertex v is the number its neighbors.

Definition 6.41 (r-regular graph)**.** G is regular of degree r if every vertex has degree r. We also say that such a graph is r-regular.

Definition 6.42 (Bipartite graphs)**.** A graph is **bipartite** if V can be partitioned into two parts, $V = V_1 \dot\cup V_2$, such that edges only go between the two parts. (Vertices within each are not adjacent.)

DO 6.43. Prove: G is bipartite if and only if its vertices can be colored by two colors, say red and blue, such that no two vertices of the same color are adjacent (neighbors have different color).

DO 6.44. Show that C_n is bipartite if and only if $2 \mid n$.

DO 6.45 (Characterization of bipartite graphs)**.** Show that G is bipartite if and only if G does not contain any odd cycles.

Definition 6.46 (Forest)**.** G is a *forest* if G contains no cycles.

Definition 6.47. w is accessible from v if there exists a path from v to w.

DO 6.48. Show that accessibility is an equivalence relation on V.

Definition 6.49 (Connected components)**.** The equivalence classes of the accessibility relation are called **connected components**.

Definition 6.50 (Connected graph)**.** G is connected if it has just one connected component, i.e., every vertex is acessible from every vertex.

Definition 6.51 (Tree)**.** A *tree* is a connected forest, i.e., it is a connected graph with no cycles.

HW 6.52. Draw all non-isomorphic trees on 7 vertices (by hand). State how many you found. Do it in some systematic way (explain your system). Avoid either of the following types of mistakes: draw two graphs that are isomorphic; miss a graph; draw a graph that is not a tree or has the wrong number of vertices. **(11 points minus 3 points per mistake)**

Definition 6.53. G is self-complementary if $G \cong \overline{G}$.

HW 6.54. (a) Find a self-complementary graph on 4 vertices and one on 5 vertices. Name them (their names appear in these notes); you do not need to draw.
(b) Prove: If G is self-complementary then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. **(2+5 points)**

HW 6.55. Prove: If G is bipartite then $m \leq n^2/4$. **(6 points)**

XC 6.56. Prove: If G has no triangles then $m \leq n^2/4$. **(8 points)**