# Discrete Math 37110 - Class 7 (2016-10-18)

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## 7.1 Automorphisms, Platonic solids as graphs

**Definition 7.1.** $\text{Iso}(G, H)$ is the set of all isomorphisms from $G$ to $H$.
An *automorphism* of $G$ is a $G \to G$ isomorphism.
$\text{Aut}(G) = \text{Iso}(G, G)$ is the set (in fact, group) of automorphisms.

**DO 7.2.** Show that $|\text{Aut}(C_n)| = 2n$

**DO 7.3.** Prove: $|\text{Aut}(G)| = n! \iff G = K_n$ or $G = \overline{K_n}$.

**Definition 7.4** (Complete bipartite graph)**.** We define $K_{r,s}$ as a graph with $r + s$ vertices divided into a group of $r$ vertices and a group of $s$ vertices, and each vertex in one group is adjacent to each vertex in the other group.

**Fact 7.5.** $|V(K_{r,s})| = r + s$ and $|E(K_{r,s})| = r \cdot s$.
$\text{Aut}(K_{r,s}) = r!s!$ if $r \neq s$, and $|\text{Aut}(K_{r,s})| = 2(r!)^2$ if $r = s$.

**DO 7.6.** Study the five Platonic solids: tetrahedron, octahedron, cube, dodecahedron, icosahedron.

We can view polyhedra such as the Platonic solids as graphs (they have vertices and edges).

The vertices and edges of the *tetrahedron* form the graph $K_4$, with $4! = 24$ automorphisms.

**DO 7.7.** Prove: The *cube* (as a graph) has $8 \cdot 6 = 48$ automorphisms.

**DO 7.8.** Prove: The *octahedron* has 48 automorphisms.

Hint. The octahedron is a 4-regular graph with 6 vertices. Its complement is a 1-regular graph (i.e., a "perfect matching") consisting of 3 disjoint edges, so it has $3! \cdot 2^3 = 48$ automorphisms.

**DO 7.9.** Prove that the automorphism group of the cube and the automorphism group of the octahedron are isomorphic.

Hint. The octahedron and the cube are dual graphs. (We shall learn more about dual graphs when we study planar graphs.)

The *dodecahedron* has 12 faces, 30 edges, 20 vertices.
The *icosahedron* has 12 vertices, 30edges, 20 faces.
They are dual to each other.

**DO 7.10.** The dodecahedron has 120 automorphisms.

(Therefore the icosahedron also has 120 automorphisms.)

**CH 7.11.** Show that Aut(cube) $\cong S_4 \times C_2$.

**CH 7.12.** Show that Aut(dodecahedron) $\ncong S_5$

**Definition 7.13.** The *girth* of a graph is the length of its shortest cycle.

**Definition 7.14** (Petersen's graph)**.** Petersen's graph has 10 vertices, is 3-regular, and has girth 5.

**DO 7.15.** Prove that these properties uniquely define a graph. Prove that this graph (i.e., Petersen's graph) can be obtained from the dodecahedron by identifying opposite vertices.

**DO 7.16.** (a) Show that the dodecahedron is Hamiltonian.
(b) Show: Petersen's graph is not Hamiltonian. (I don't know any elegant proof of this fact.)

**XC 7.17.** Show that $|\text{Aut(Petersen's)}| = 120$. **(6 points)**

**CH 7.18.** Show that Aut(Petersen's) $\cong S_5$.

**CH 7.19.** Show that Aut(dodecahedron) $\cong A_5 \times C_2$ where $A_5$ denotes the alternating group of degree 5 (the group of even permutations of 5 elements).

## 7.2 Trees

**Definition 7.20** (Tree)**.** A *tree* is a connected cycle-free graph.

**Theorem 7.21.** *For a tree,* $m = n - 1$*.*

**Lemma 7.22.** *If a tree has* $n \geq 2$ *vertices, it has a vertex of degree* 1 *(a "dangling vertex").*

**DO 7.23.** Show that the endpoints of a maximal path in a tree with $n \geq 2$ vertices are dangling.

*Proof of theorem 7.21.* By induction on $n$. Base case: If $n = 1$, $m = 0$.

Assume now $n \geq 2$. Inductive Hypothesis: Theorem true for all trees with $n' < n$ vertices.

By the Lemma, there exists a vertex $x$ of degree 1. Let $T' = T \setminus x$ (the subgraph iduced on the vertices other than $x$).

**Claim.** $T'$ is a tree.

Proof. $T'$ is clearly cycle-free. We need to show $T'$ is connected.

Let $u, v \in V(T')$. Then there exists a path $P$ in $T$ from $u$ to $v$. Clearly, $x$ cannot be an endpoint of $P$ since $x$ is not in $T'$. Moreover, $x$ cannot be an interior point of $P$ because it would need degree $\geq 2$ to be an the interior point of a path. This proves the Claim.

So $T'$ is a tree and therefore we can apply the IH to $T'$. We have $n' = n - 1$, and $m' = m - 1'$, and by the IH, $m' = n' - 1$. So $m - 1 = (n - 1) - 1$, and therefore $m = n - 1$. $\square$

**DO 7.24.** For a graph $G$ with $n$ vertices and $m$ edges, the following are equivalent:

(1) $G$ is a tree

(2) $G$ is connected and $m = n - 1$

(3) $G$ is cycle-free and $m = n - 1$

(4) $(\forall u, v \in V)(\exists! u - \cdot - v \text{ path})$

(5) $G$ is a maximal cycle-free graph

(6) $G$ is a minimal connected graph

**DO 7.25.** Every connected graph has a spanning tree

**Example 7.26.** The number of spanning trees on $K_n$ = trees on a given set of $n$ vertices:
$n = 1 : 1$
$n = 2 : 1$
$n = 3 : 3$
$n = 4 : 16$ (12 copies of $P_4$ and 4 copies of $K_{1,3}$)
$n = 5 : 125$ (5 copies of $K_{1,4}$, 60 copies of $P_5$, and 60 copies of a Y-shaped tree)

**DO 7.27.** Let $T$ be a tree on $n$ vertices. Prove: the number of copies of $T$ in $K_n$ is

$$\frac{n!}{|\text{Aut}(T)|}.$$

**Theorem 7.28** (Cayley's Formula)**.** *$K_n$ has $n^{n-2}$ spanning trees. In other words, there are $n^{n-2}$ trees on a given set of $n$ vertices.*

## 7.3 Independent sets, chromatic number

**Definition 7.29** (Independent set)**.** Let $G = (V, E)$ be a graph. We say that a subset $A \subseteq V$ is *independent* in $G$ if the induced subgraph $G[A]$ is empty (there are no edges among $A$).

**Definition 7.30** (Independence number)**.** $\alpha(G)$ is the maximum size of independent sets.

**DO 7.31.** Find $\alpha(C_n)$, $\alpha(P_n)$, $\alpha(K_n)$, $\alpha(K_{r,s})$.

**Definition 7.32** (Legal Coloring)**.** A coloring $f : V \to \{\text{colors}\}$ of the vertices of $G$ is *legal* if neighbors always get different colors, i.e., if $v \sim w$ then $f(v) \neq f(w)$. The *chromatic number* $\chi(G)$ is the smallest number of colors in a legal coloring.

**HW 7.33.** Show that $\alpha(G)\chi(G) \geq n$ **(7 points)**

**HW 7.34.** Show that $(\forall d)($ if $(\forall v \in V)(\deg(v) \leq d)$, then $\chi(G) \leq d + 1)$. **(6 points)**