Discrete Math 37110 - Class 10 (2016-10-27)

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10.1 Variance, Chebyshev's Inequality, Law of Large Numbers

A random variable X is a function from the sample space Ω to $\mathbb{R}$

The expected value of X is $E(X) = \sum_{a\in\Omega} P(a)X(a) = \sum_{y\in\mathbb{R}} yP(X = y)$

Definition 10.1. X is a non-negative random variable if $(\forall a \in \Omega)(X(a) \geq 0)$

Theorem 10.2 (Markov's Inequality)**.** *If X is a non-negative random variable, then*

$$(\forall b > 0)\left(P(X \geq b) \leq \frac{E(X)}{b}\right)$$

Proof.

$$E(X) = \sum_{a\in\Omega} P(a)X(a) \geq \sum_{a\in\Omega\wedge X(a)\geq b} P(a)X(a) \geq b \sum_{a\in\Omega\wedge X(a)\geq b} P(a) = bP(X \geq b)$$

Since b is positive, we then get $P(X \geq b) \leq \frac{E(X)}{b}$ □

Theorem 10.3 (Chebyshev's Inequality)**.** *For any random variable X,*

$$P(|X - E(X)| \geq d) \leq \frac{\mathrm{Var}(X)}{d^2}$$

Proof. Let $m = E(X)$ and $Y = (X-m)^2 \geq 0$. The left hand side is equivalent to $P(Y \geq d^2)$. By Markov's Inequality, $P(Y \geq d^2) \leq \frac{E(Y)}{d^2} = \frac{\mathrm{Var}(X)}{d^2}$ □

Recall: if $X = Y_1 + \cdots + Y_k$, then

$$\mathrm{Var}(X) = \sum_{i=1}^{k}\sum_{j=1}^{k} \mathrm{Cov}(Y_i, Y_j).$$

Corollary 10.4. *If $Y_1, \ldots, Y_k$ are pairwise uncorrelated, then* $\mathrm{Var}(\sum Y_i) = \sum \mathrm{Var}(Y_i)$

Corollary 10.5. *If $Y_1, \ldots, Y_k$ are pairwise independent, then* $\mathrm{Var}(\sum Y_i) = \sum \mathrm{Var}(Y_i)$

Property	Assumption
Linearity of expectations	no assumption
Additivity of variance	pairwise independence (uncorrelated is sufficient)
Multiplicativity of expectation	(fully) independent

Definition 10.6. The *distribution* of a random variable X is the function $f_X : \mathbb{R} \to \mathbb{R}$ defined by $f_X(y) = P(X = y) \quad (y \in \mathbb{R})$. (This definition only works for finite sample spaces or more generally, for discrete random variables.)

Example 10.7. For the binomial distribution (number of successes in a sequence of n independent Bernoulli trials), $f_{X_n}(x) = \binom{n}{y} p^y (1-p)^{n-y}$.

DO 10.8. Show that $\mathrm{Var}(\alpha X) = \alpha^2 \mathrm{Var}(X)$ for all $\alpha \in \mathbb{R}$.

Definition 10.9. X and Y are identically distributed if $f_X = f_Y$. A sequence of "i.i.d. random variables" means a sequence of independent, identically distributed random variables.

Theorem 10.10 (Baby law of Large Numbers)**.** *Let X_n be the number of successes in a sequence of n i.i.d. Bernoulli trials with success probability p. For every (fixed) $\epsilon > 0$, the probability $P\left(\left|\frac{X_n}{n} - p\right| \geq \epsilon\right)$ approaches zero es $n \to \infty$.*

Proof.

$$P\left(\left|\frac{X_n}{n} - p\right| \geq \epsilon\right) \leq \frac{\mathrm{Var}(\frac{X_n}{n})}{\epsilon^2} = \frac{\mathrm{Var}(X_n)}{n^2\epsilon^2} = \frac{np(1-p)}{n^2\epsilon} \leq \frac{1}{4n\epsilon^2} \to 0.$$

□

10.2 Directed graphs, strong components, cut, tournaments

DO 10.11. Check graph and digraph terminology in LN.

Definition 10.12 (Directed graph/digraph)**.** $G = (V, E)$ where $E \subseteq V \times V$.

We may put a little arrow on top of the G to emphasize that we are talking about digraphs, but this is not mandatory: $\vec{G} = (V, \vec{E})$

Definition 10.13 (Directed Path)**.** $\vec{P}_n$ is the directed path of length $n-1$ (with n vertices). Here $n \geq 1$.

Definition 10.14 (Directed Cycle)**.** $\vec{C}_n$ is the directed cycle with n vertices ($n \geq 1$). A cycle with just one vertex is called a (self)-loop.

Definition 10.15. Given $v, w \in V$, we say that w is accessible from v if there exists a (directed) path from v to w.

DO 10.16. Prove: accessibility is a (a) reflexive and (b) transitive relation on V.

Hint. (a) Use paths of length 0 $\quad \vec{P}_1$
(b) Use the next exercise.

DO 10.17. Prove that if there exists a walk from u to w, then there exists a path from u to w. (Walks may repeat vertices and edges.)

DO 10.18. Show that mutual accessibility is an equivalence relation.

Definition 10.19. The equivalence classes of the mutual accessibility relation are called the **strong components** of G.

Definition 10.20. (A, B) is a cut of G if $V = A\dot{\cup}B$ (disjoint union) and $A, B \neq \emptyset$, and $E(B, A) = \emptyset$ where $E(A, B)$ is the set of edges from B to A

Definition 10.21. A digraph G is *strongly connected* if it has just one strong component; in other words, $(\forall u, w \in V)(w$ is acessible from $v)$

HW 10.22. A digraph G is strongly connected if and only if there does not exist a cut. **(7 points)**

Definition 10.23. G digraph is weakly connected if $\widetilde{G}$, the undirected version of G, is connected.

Definition 10.24. A digraph G is Eulerian if $(\forall v \in V)(\deg^+(v) = \deg^-(v))$ (where $\deg^+$ is the out-degree and $\deg^-$ is the in-degree)

XC 10.25. If $\vec{G}$ is Eulerian and weakly connected, then it is strongly connected.
(5 points)

Definition 10.26. Given an undirected graph G, $\vec{G}$ is an orientation of G if we assign an direction to each edge. The number of orientations of G is 2^m

Definition 10.27. An *oriented graph* is a digraph without cycles of length 1 or 2

Definition 10.28. A *tournament* is an orientation of K_n

DO 10.29. The number of tournments on a given set of n vertices is $2^{\binom{n}{2}}$.

DO 10.30. Prove that every tournament has a Hamilton path.

XC 10.31. Prove that every strongly connected tournament has a Hamilton cycle.
(5 points)

DO 10.32. Find a strongly connected digraph that is not Hamiltonian (has no Hamilton cycle).

Definition 10.33. A legal coloring on a digraph is a coloring c of the vertices such that that $(i, j) \in E \implies c(i) \neq c(j)$. The chromatic number of a digraph G is the minimum number of colors in a legal coloring of G. So if G has no loops then $\chi(G) = \chi(\widetilde{G})$.

XC 10.34. (a) If $(\forall v \in V)(\deg^+ \leq k)$, then $\chi(\vec{G}) \leq 2k + 1$.

(b) Prove that this bound is tight for every $k \geq 1$. **(6+3 points)**