# Discrete Math 37110 - Class 11 (2016-11-01)

Instructor: László Babai
Notes taken by Jacob Burroughs
Revised by instructor

## 11.1 Winning strategy

**Example 11.1** (Divisor game)**.** Fix $n \geq 2$. Two players alternate picking positive divisors of $n$. No player can pick a divisor of a previously selected number. Whoever is forced to pick $n$ loses.

**CH 11.2.** The first player has a winning strategy.

## 11.2 Ramsey Theory, Erdős's Probabilistic Method

**Example 11.3** (Ramsey game)**.** Given a set of 6 points, two players, "red" and "blue," alternate joining pairs of points by a line of their color. The first player to make a triangle in their color loses.

**Notation 11.4** (Erdős–Rado arrow symbol)**.** $n \to (k, \ell)$ if for all colorings of $E(K_n) \to \{\text{red}, \text{blue}\}$ either $\exists \text{red} K_k$ or $\exists \text{blue} K_\ell$. Notation for the diagonal case: If $n \to (k, k)$, we denote this circumstance by $n \to (k)_2$.

**DO 11.5.** $6 \to (3, 3)$

**DO 11.6.** $10 \to (4, 3)$

**CH 11.7.** $9 \to (4, 3)$

**DO 11.8.** (a) Define $n \to (k, \ell, m)$

(b) Prove $17 \to (3, 3, 3)$

**Theorem 11.9** (Erdős-Szekeres)**.** $\binom{k+\ell}{k} \to (k+1, \ell+1)$

**DO 11.10.** Prove this theorem by induction on $k + \ell$.
Hint. The inductive step: reduce $(k, \ell)$ to $(k-1, \ell)$ and $(k, \ell-1)$, using Pascal's Identity. Base case: $k = 1$ or $\ell = 1$. If $k = 1$, we have $\binom{1+\ell}{1} = 1 + \ell \to (2, \ell+1)$. Same for $\ell = 1$.

**DO 11.11.** Read bio of Frank Plumpton Ramsey

**Theorem 11.12** (Ramsey's Theorem (special case))**.** $(\forall k_1, \ldots, k_r)(\exists n)(n \to (k_1, \ldots, k_r))$

**Remark 11.13.** This case is special in that it deals with partitions of $\binom{V}{2}$. The general case deals with partitions of $\binom{V}{s}$ for any fixed $s$.

**CH 11.14.** Prove Ramsey's Theorem for triples (we color $\binom{V}{3}$).

**Remark 11.15** (Ramsey numbers)**.** Define $R(k)$ to be the smallest $n$ such that $n \to (k,k)$. $R(4)$ is known; $R(5)$ is an open problem, and $R(6)$ is not believed to be solvable.

**DO 11.16.** If $n' > n \to (k,\ell)$, then $n' \to (k,\ell)$

We observe that $n \to (\frac{1}{2}\log_2 n)_2$

**DO 11.17.** Prove $n \not\to (\sqrt{n}+1)_2$ by an explicit example. You may assume $n = k^2$.

If $n \to (k,\ell)$, $\forall$ graphs $G$ with $n$ vertices, either the clique number $\omega(G) \geq k$ or the independence number $\alpha(G) \geq \ell$

On the flip side of the observation above, $n \not\to (2\log_2 n + 1)_2$

*Proof.*

**Lemma 11.18.** *For almost all graphs $G$, $\alpha(G) \leq 2\log_2 n + 1$*

Let $A \subseteq [n] = V$. Let $|A| = k$

Then $P(G[A] = \emptyset) = \frac{1}{2^{\binom{k}{2}}}$

$P(\exists A \subseteq V, |A| = k, \emptyset) \leq \binom{n}{k}\frac{1}{2^{\binom{k}{2}}}$, by union bound

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} < \frac{1}{k!}\frac{n^k}{2^{\frac{k(k-1)}{2}}} = \frac{1}{k!}\left(\frac{n}{2^{\frac{k-1}{2}}}\right)^k$$

In order to show that the right-hand side approaches zero, it suffices to show that $\frac{n}{2^{\frac{k-1}{2}}} \leq 1$, which is equivalent to $1 + 2\log_2 n \leq k$. □

**Corollary 11.19.** *For almost all graphs, there is no homogeneous subset of size $1 + 2\log_2 n$*

**Corollary 11.20.** $n \not\to (1 + 2\log_2 n)_2$ *for sufficiently large $n$*

**Remark 11.21.** An explicit construction for the above corollary, or anything close, has not been found.

**Remark 11.22.** The **Probabilistic Method** proves the existence of objects with certain properties without constructing such objects. The method proves existence by constructing a probability space and proving that random sampling will find an object with the desired properties with positive probability. Often (as above), the object is found with high porbability. Yet, often an explicit construction of such objects is extremely difficult.

**CH 11.23.** $(\forall k)(\exists G)(G \not\supset K_3, \chi(G) \geq k)$

**DO 11.24.** For almost all graphs, $\chi(G) \geq (\omega(G))^{100}$.

## 11.3 Planar graphs

**Definition 11.25.** $G$ is *planar* if there exists a plane drawing of $G$ without intersections of edges.

**Example 11.26.** $K_4$ is planar.

**Theorem 11.27.** *$K_5$ and $K_{3,3}$ are not planar.*

**DO 11.28.** Infer this theorem from the following two results.

**Theorem 11.29.** *If $G$ is a planar graph with $n \geq 3$, then $m \leq 3n - 6$.*

**Theorem 11.30.** *If $G$ is planar and triangle-free with $n \geq 3$, then $m \leq 2n - 4$.*

**Example 11.31.**

| | n | m | f | n - m + f |
|---|---|---|---|---|
| Tetrahedron | 4 | 6 | 4 | 2 |
| Cube | 8 | 12 | 6 | 2 |
| Octahedron | 6 | 12 | 8 | 2 |
| Dodecahedron | 20 | 30 | 12 | 2 |
| Isocahedron | 12 | 30 | 20 | 2 |

**Theorem 11.32** (Euler's formula)**.** *If $G$ is connected, planar then $n - m + f = 2$.*