Discrete Math 37110 - Class 12 (2016-11-03)

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12.1 Monotone subsequences, the Pigeonhole Principle

Given $a_1, a_2, \dots, a_n \in \mathbb{R}$, we call $a_{i_1} < a_{i_2} < \cdots a_{i_r}$ where $i_1 < i_2 < \cdots < i_r$ an increasing subsequence, and the reverse a decreasing subsequence.

Theorem 12.1 (Erdős-Szekeres)**.** *If $n = k\ell + 1$ then every sequence of length n of distinct real numbers has an increasing subsequence of length $k+1$ or a decreasing subsequence of length $\ell + 1$.*

Proof. Assume this is false. Let x_i denote the length of the longest increasing subsequence of which a_i is the last term, and let y_i denote the length of the longest decreasing subsequence of which a_i is the last term. Then for a given a_i and corresponding (x_i, y_i) $1 \le x_i \le k$ and $1 \le y_i \le \ell$. So we only have $k\ell$ options for the the pair (x_i, y_i). But $n > k\ell$, so, by the pigeon-hole principle, $(x_i, y_i) = (x_j, x_j)$ for some $i < j$. But if $a_i < a_j$, then $x_i < x_j$ and if $a_i > a_j$ then $y_i < y_j$. This gives a contradiction, proving the Erdös-Szekeres theorem. □

Corollary 12.2. *If $n = k^2 + 1$, there exists a monotone subsequence of length $\ge k + 1$.*

12.2 Inclusion-Exclusion

Let $A_1, \dots, A_k \subseteq \Omega$ and $B = \overline{A_1 \cup \cdots \cup A_k}$

Given $P(A_i), P(A_i \cap A_j), P(A_i \cap A_j \cap A_\ell)$, etc., how do we find $P(B)$?

Theorem 12.3 (Inclusion–Exclusion)**.** *$P(B) = S_0 - S_1 + S_2 - \cdots$, where $S_0 = 1$, and $S_j = \sum_{i_1 < i_2 < \cdots < i_j} P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j})$.*

Proof. Let $x \in \Omega$. Let $r(x) = |\{i \mid x \in A_i\}|$

x was counted k_x times: $k_x = \binom{r}{0} - \binom{r}{1} + \binom{r}{2} + \cdots = (1-1)^r = 0^r = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases}$ □

Theorem 12.4 (Restatement of the Inclusion-Exclusion formula)**.**

$$P(B) = \sum_{I \subseteq [n]} (-1)^{|I|} P\left(\bigcap_{i \in I} A_i\right).$$

DO 12.5. Show that Theorems 12.3 and 12.4 are equivalent.

We define the indicator of the set $A \subseteq \Omega$ by setting $\theta_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ for $x \in \Omega$.

DO 12.6. Show that $\theta_A \theta_B = \theta_{A\cap B}$

DO 12.7. Show that $\theta_{A\cup B} = \theta_A + \theta_B - \theta_A\theta_B$

DO 12.8.

$$(1+x_1)(1+x_2)\cdots(1+x_k) = \sum_{I\subseteq[k]} \prod_{i\in I} x_i$$

$$(1-x_1)(1-x_2)\cdots(1-x_k) = \sum_{I\subseteq[k]} (-1)^{|I|} \prod_{i\in I} x_i$$

Alterantive proof of Inclusion–Exclusion. We find that

$$\theta_B = \prod \theta_{\overline{A_i}} = \prod(1-\theta_{A_i}) = \sum_{I\subseteq[k]} (-1)^{|I|}\theta_{\bigcap_{i\in I} A_i}.$$

Therefore

$$P(B) = E(\theta_B) = \sum_{I\subseteq[k]} (-1)^{|I|} E(\theta_{\bigcap_{i\in I} A_i}) == \sum_{I\subseteq[k]} (-1)^{|I|} P\left(\bigcap_{i\in I} A_i\right).$$

$\square$

Definition 12.9. A *derangement* of a set A is a fixed-point-free permutation of A.

DO 12.10. Let d_n denote the probability that a random permutation of the set $[n]$ is a derangement. Prove: $\lim_{n\to\infty} d_n = 1/\mathrm{e}$. In fact, $\left|d_n - \frac{1}{\mathrm{e}}\right| < \frac{1}{(n+1)!}$.

DO 12.11. Bonferroni's inequalities:

$$\begin{aligned} P(B) &\le S_0 \\ P(B) &\ge S_0 - S_1 \\ P(B) &\le S_0 - S_1 + S_2 \\ P(B) &\ge S_0 - S_1 + S_2 - S_3 \\ &\vdots \end{aligned}$$

12.3 Planarity, multigraphs, Euler's formula

Definition 12.12. A *multigraph* $G = (V, E, f)$ consists of a set V of vertices, a set E of edges, and a map $f : E \to V \cup \binom{V}{2}$; this map defines the two endpoints of the edge. If the two endpoints are the same, we say that the edge is a *loop*. Note that multiple edges can have the same set of endpoints.

Definition 12.13. A *simple arc* is the range of a continuous injection $f : [0,1] \to \mathbb{R}^2$ of the $[0,1]$ segment into the plane. A *Jordan curve* is the range of a continuous injection of the unit circle in the plane, or equivalently, the range of a continuous map $f : [0,1] \to \mathbb{R}^2$ such that $f(x) = f(y) \iff x = y$ or $\{x, y\} = \{0, 1\}$.

Definition 12.14. Given a multigraph G, a plane embedding $\widetilde{G}$ of G associates with every vertex a point of the plane and with every edge a simple arc between its endpoints so that those arcs do not intersect except in their shared vertices.

Definition 12.15. A multigraph G is planar if there exists a plane embedding of G. A *plane (multi)graph* is a (multi)graph embedded in the plane.

Definition 12.16. We define a *face* of a plane graph $\widetilde{G}$ as a connected component of $\mathbb{R}^2 \setminus \tilde{G}$. (The connected components are the equivalence classes of $\mathbb{R}^2 \setminus \tilde{G}$ under the relation "equal or accessibile by a simple arc.")

Theorem 12.17 (Jordan curve Theorem)**.** *A Jordan curve has two faces.*

The ≤ 2 part is easy but proving ≥ 2 is suprisingly hard.

Theorem 12.18 (Euler's formula)**.** *If G is a connected multigraph and $\widetilde{G}$ a plane embedding of G then $\widetilde{(G)}$ satisfies $n - m + f = 2$.*

Remark 12.19. Note that if G is a cycle then Euler's formula is equivalent to the Jordan curve theorem.

DO 12.20. Prove: a plane embedding of a tree has one face. Hint: induction on n.

DO 12.21. Prove Euler's formula by induction on m, using the case of trees as the base case. Note where you use the Jordan curve theorem.