Discrete Math 37110 - Class 14 (2016-11-10)

Instructor: László Babai
Notes taken by Jacob Burroughs
Revised by instructor

14.1 Plane graphs, duality

Definition 14.1. A *plane graph* is representation/drawing of a graph in the plane (or on the sphere).

Definition 14.2. G is *planar* if there exists a plane drawing of G.

Definition 14.3. The *dual* of a plane multigraph graph is the plane multigraph where every face of the original multigraph becomes a vertex, and an edge connects any vertices whose corresponding faces share an edge in the original graph.

Example 14.4. The dual of the cube ($n = 8, m = 12, f = 6$) is the octahedron ($n = 6, m = 12, f = 8$).

In general, $n_{\text{dual}} = f$, $m_{\text{dual}} = m$, and $f_{\text{dual}} = n$

DO 14.5. The dual of a dual is the original multigraph.

HW 14.6. (a) Draw two plane multigraphs that are isomorphic as multigraphs but have non-isomorphic duals. Make the number of edges as small as possible. You do not need to prove minimality.
(b) Solve the same problem with graphs (not multigraphs). Neither the graph nor either of the two non-isomorphic duals should have multiple edges or loops. Again, make the number of edges as small as possible. You do not need to prove minimality.
(5+4 points)

Definition 14.7. Number of sides of a face: if the face touches an edge from both sides, then that edge counts twices towards the number of sides. For instance a tree with n vertices has 1 face, and that face has $2n - 2$ sides.

Theorem 14.8 (Handshake theorem for faces)**.** *The sum of the number of sides of the faces of a graph is $2m$.*

Definition 14.9. A graph with $n \geq 2$ vertices is *k-connected* if for all $v \neq w \in V$ there exist k internally disjoint paths from v to w. (The paths don't share interior vertices, only their endpoints are shared.)

Theorem 14.10 (Whitney)**.** *If k is a 3-connected planar graph then its drawing on the sphere is unique (up to natural equivalence and reflection).*

Theorem 14.11. *The vertices and edges of convex polyhedron form a 3-connected planar graph.*

Theorem 14.12 (Mani)**.** *Every 3-connected planar graph occurs this way (as the vertices and edges of a convex polyhedron).*

14.2 Permutations, determinants

Definition 14.13. $M_n(\mathbb{R})$ is the set of all real $n \times n$ matrices.
$M_n(\mathbb{Z})$ is the set of all integral $n \times n$ matrices.

Definition 14.14 (Determinant)**.** Let $A \in M_n(\mathbb{R})$. Then

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

S_n is the set of all permutations of $[n]$. Given $\sigma \in S_n$, $i \neq j \in [n]$ are in inversion if $(i-j)(\sigma(i)-\sigma(j)) < 0$. Then we define $\operatorname{Inv}(\sigma)$ to be the number of pairs $\{i, j\}$ in inversion.

DO 14.15. $0 \leq \operatorname{Inv}(\sigma) \leq \binom{n}{2}$.

DO 14.16. $\operatorname{Inv}(\sigma) = 0 \iff \sigma = \mathrm{id}$

DO 14.17. $\operatorname{Inv}(\sigma)(\sigma) = \binom{n}{2} \iff \sigma = \text{reversal}$

Definition 14.18. $\operatorname{sgn}(\sigma) = (-1)^{\operatorname{Inv}(\sigma)}$. We say that a σ is an *even permutation* if $\operatorname{sgn}(\sigma) = 1$ and an *odd permutation* if $\operatorname{sgn}(\sigma) = -1$.

We define multiplication (composition) of permutations left-to-right: $\sigma\tau(i) = \tau(\sigma(i))$.
Every permutation can be uniquely represented by directed graph consisting of disjoint directed cycles.
A *cyclic permutation* is a permutation with a single cycle of length > 1 (the other points are fixed). A cycle of length k is denoted as $(a_1\ a_2\ \cdots\ a_k)$ where $\sigma(a_i) = a_{i+1}$ where the subscript is calculated mod k.)
A cycle of length 2 is called a *transposition.* A "neighbor-swap" is a transposition of the form $(i\ i+1)$.

DO 14.19. If τ is a neighbor swap then $|\operatorname{Inv}(\sigma\tau) - \operatorname{Inv}(\sigma)| = 1$ for any $\sigma \in S_n$.

DO 14.20. $\operatorname{sgn}(\sigma\tau) = -\operatorname{sgn}(\sigma)$.

DO 14.21. If $\tau_1, \ldots, \tau_k$ are neighbor swaps, then $\operatorname{sgn}(\sigma\tau_1 \ldots \tau_k) = \operatorname{sgn}(\sigma)(-1)^k$.

DO 14.22. If $\tau_1, \ldots, \tau_k$ are neighbor swaps, then $\operatorname{sgn}(\tau_1 \ldots \tau_k) = (-1)^k$

DO 14.23. Every permutation is a product of neighbor swaps.

DO 14.24. $\operatorname{sgn}(\sigma_1\sigma_2) = \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2)$
In other words (for those who studied group theory), sgn is a homomorphism from the group S_n to the multiplicative group $\{1, -1\}$.

DO 14.25. The transposition $(i\ j)$ is the product of $2(j-i)-1$ neighbor swaps.

DO 14.26. Every transposition is an odd permutation.

DO 14.27. If $\sigma = \tau_1 \cdots \tau_k$ where the τ_i are transpositions, then $\operatorname{sgn}(\sigma) = (-1)^k$.

DO 14.28. Prove: for $n \geq 2$, exacly half of the permutations is even and half is odd.

DO 14.29. Study "Sam Lloyd's 15 puzzle." Prove: exctly half of the configurations is solvable.

DO 14.30 (Laplace expansion by row i)**.** $\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$ where $C_{ij} = (-1)^{i+j} \det(A_{\hat{i}\hat{j}})$ where $A_{\hat{i}\hat{j}})$ is the $(n-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

HW 14.31. Find $\det(T_n)$ as a closed-form expression where T_n is the $n \times n$ tri-diagnal matrix with entries $a_{ii} = 1 = a_{i,i+1} = 1$ and $a_{i,i-1} = -1$; all other entries are zero.

DO 14.32. If a row is all zeros, then $\det(A) = 0$

DO 14.33. Show that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$ and in fact $\operatorname{Inv}(\sigma) = \operatorname{Inv}(\sigma^{-1})$.

Definition 14.34. The *transpose* of the matrix $A = (a_{ij})$ is the matrix $A^T = (b_{ij})$ where $b_{ij} = a_{ji}$.

DO 14.35. $\det(A) = \det(A^T)$.

DO 14.36. If we swap two columns of A to make A', then $\det(A') = -\det(A)$.
More generally, let A^σ be A with σ applied to the columns. Then $\det(A)^\sigma = \det(A)\operatorname{sgn}(\sigma)$.

DO 14.37. If two columns of A are equal then $\det(A) = 0$.

Definition 14.38 (Elementary column operation)**.** Let $A = [a_1, \dots, a_n]$ be a matrix with columns $a_1, \dots, a_n$. An elementary column operation on A is defined by the triple (i, j, λ) where $i, j \in [n]$; the result is the new matrix A' with columns $a'_1, \dots, a'_n$ where $a'_i = a_i - \lambda a_j$ and $a'_k = a_k$ for all $k \neq i$.

DO 14.39. If A' is obtained from A be an elementary column operation then $\det(A') = \det(A)$.

DO 14.40. If the columns of A are linearly dependent then $\det(A) = 0$.

DO 14.41. The converse holds as well, i.e., if $\det(A) = 0$ then the columns of A are linearly dependent.

Notation 14.42. The *identity matrix* $I = I_n$ is the $n \times n$ matrix with $I = (\delta_{ij})$ where $\delta_{ij} = 1$ if $i = j$ (diagonal) and $\delta_{ij} = 0$ if $i \neq j$ ("Kronecker delta"). All entires of the all-ones matrix J are equal to 1.

HW 14.43. Let $A = aI + b(J - I)$ (all diagonal entries are equal to a and all off-diagonal entries are equal to b). Find a closed-form expression of $\det(A)$ in terms of a, b, n.