Discrete Math 37110 - Class 15 (2016-11-15)

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Partially revised by instructor 11/16 3:35pm. Errors still possible. HW statements have been revised; the first problem in the earlier posting was misstated. Compare exercises with LN and Lin Alg. text and your own notes. Report errors in HW statements to instructor.

15.1 Planar graphs, plane graphs, Kuratowski's Theorem

DO 15.1. The only way to get a one-sided face is through a self-loop. One can get a 2-sided face from a single edge ($n = 2$) or two parallel edges.

DO 15.2. If $n \geq 3$ for a connected plane graph, there are no 1 or 2 sized faces. (Here we are talking about a graph, not a multigraph. To emphasize this, we could add the adjective "simple" — so a "simple graph" just means a graph.)

HW 15.3 (revised)**.** Draw a connected plane multigraph which has a pair of parallel edges but no loops, no bridges, and no 2-sided face. Use as few edges as you can. (Recall: a bridge is an edge whose removal makes the graph disconnected. By removing the edge we mean removing the edge from the list of edges; but we keep its endpoints as vertices.) **(5 points)**

Theorem 15.4. *If G is a planar graph where $n \geq 3$, then $m \leq 3n - 6$*

Proof. Let f_k be the number of k-sided faces. The sum of the number of sides of each face $\sum_{u \text{ is a vertex of the dual}} \deg(u) = \sum_k k f_k = 2m$. Since $n \geq 3$, $f_1 = f_2 = 0$ Then $2m = \sum_{k=3}^{\infty} k f_k \geq 3 \sum f_k = 3f$. Therefore $2m \geq 3f$. Applying Euler's formula ($n - m + f = 2$): $2m \geq 3(2 - n + m)$, or equivalently, $m \leq 3m - 6$ □

DO 15.5. If G is a triangle-free planar graph where $n \geq 3$, then $m \leq 2n - 4$

Definition 15.6. The *girth* is the length of the shortest cycle.

DO 15.7. The maximum girth of a connected planar graph where every vertex has degree ≥ 3

A subdivision of K_5 is a topological K_5, and is also non-planar.
The *Kuratowski* graphs are topological K_5 and topological $K_{3,3}$

Theorem 15.8 (Kuratowski's Theorem)**.** *G is planar if and only if G does not contain a Kuratowski graph.*

HW 15.9. Find a Kuratowski subgraph in the Petersen graph. Hand in a drawing in which you highlight those vertices of the Kuratowski subgraph which come from the original K_5 or $K_{3,3}$, and highlight the paths connecting them in the Kuratowksi subgraph. **(5 points)**

CH 15.10. a. If $G \not\supset C_4$ then $m = O(n^{3/2})$

b. This bound is tight

15.2 Linear independence, eigenvalues, eigenvectors, subspaces

DO 15.11. Review matrix multiplication.

Notation 15.12. If $v \in \mathbb{R}^n$ and $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$A = (a_{ij})_{n\times n} \in M_n(\mathbb{R})$

$Av = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ where $w_i = a_i \cdot v$ where a_i is the ith row of A and the dot denotes the standard dot product: for vectors u, v we write $u \cdot v = \sum u_i v_i$.

Definition 15.13. v is an *eigenvector* of A with eigenvalue λ if

(a) $v \neq 0$

(b) $Av = \lambda v$

Definition 15.14. λ is an eigenvalue of A if $(\exists v \neq 0)(Av = \lambda v)$

Example 15.15. For the identity matrix I, since $Iv = v$, all $v \neq 0$ are eigenvectors with eigenvalue 1. The spectrum, which is the set of eigenvalues is $\{1\}$

DO 15.16. On $\begin{bmatrix} 1 & * & \dots & * & * \\ 0 & 1 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 1 & * \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$, prove that 1 is an eigenvalue and only 1 is an eigenvalue. (The asterisks indicate arbitrary numbers.)

Example 15.17. On the all-zero matrix, $v \neq 0$ are eigenvectors with eigenvalue 0. The spectrum is $\{0\}$

Definition 15.18. A is a *scalar matrix* if $A = \lambda I$

DO 15.19. If $\forall v \neq 0$ is an eigenvector of A, then A is a scalar matrix

HW 15.20. Prove: if $v_1, \dots, v_k \in \mathbb{R}^n$ are eigenvectors of A to distinct eigenvalues, then $v_1, \cdots, v_k$ are linearly independent. **(8 points)**

Definition 15.21. Linear independence of vectors: $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$ if and only if $\alpha_1 = \cdots = \alpha_k = 0$ (that is, the only linear combination of $v_1, \dots, v_k$ that is 0 is the trivial linearly combination)

To prove linear independence: suppose $\sum \alpha_i v_i = 0$ and show that $\alpha_1 = \cdots = \alpha_k = 0$

HW 15.22. Let $a_i, \ldots, a_k \in \mathbb{R}$ be distinct real numbers. Define $f(t) = \prod(t - a_i)$ and $g_j(t) = \frac{f(t)}{t-a_j} = \prod_{i,i\neq j}(t - a_i)$. (Note that the g_j are polynomials.) Prove that $g_1, \ldots, g_k$ are linearly independent. **(7 points)**

HW 15.23 (Due Tuesday 2016-11-22)**.** Let $\mathbb{R}^{\mathbb{N}}$ denote the space of infinite sequences $\underline{a} = (a_0, a_1, \ldots)$. (Notation: $\mathbb{N} = \{0, 1, 2, \ldots\}$.) In other words, $\mathbb{R}^{\mathbb{N}}$ is the space of functions $\underline{a} : \mathbb{N} \to \mathbb{R}$, identifying such a function with a sequence. Let $S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ denote the **left shift operator**, i.e., $S\underline{a} = (a_1, a_2, \ldots)$. Find the eigenvectors and eigenvalues of this operator. **(7 points)**

Our informal definition of a **vectors space** at this point is a set V on which linear combinations are defined, satisfying the usual rules. In particular, we assume that $\alpha v = 0$ if and only if $\alpha = 0$ or $v = 0$. The elements of V are "vectors" and the real numbers are "scalars."

For any set Ω, the notation $\mathbb{R}^{\Omega}$ denotes the set of $\Omega \to \mathbb{R}$ functions: $\mathbb{R}^{\Omega} = f : \Omega \to \mathbb{R}$. We define linear combinations of such functions pointwise, i.e., by setting $(\sum \alpha_i f_i)(z) = \sum \alpha_i f_i(z)$. This makes $\Omega \to \mathbb{R}$ a vector space where the "vectors" are the $\Omega \to \mathbb{R}$ functions.

Example 15.24. Random variables on a probability space form a vector space.

Definition 15.25. W is a subspace of V, denoted $W \leq V$ if $W \subseteq V$ and W is closed under linear combinations.

DO 15.26. If $W \leq V$ then $0 \in W$. In particular, $\emptyset$ is not a subspace.

DO 15.27. Characterize all subspaces of G_3, our 3D geometry with a point designated as the origin.

DO 15.28. The intersection of any number of subspaces is a subspace. This is true even if we intersect infinitely many subspaces.

Theorem 15.29. *There exists a unique smallest subspace containing $S \subseteq V$*

DO 15.30. Span(S) is the set of all linear combinations of all finite subsets of S.

HW 15.31 (Due 2016-11-22)**.** As before, $\mathbb{R}^{\mathbb{N}}$ denotes the set of infinite sequences $\underline{a} = \{(a_1, a_2, \ldots)\}$.

Let $F \subseteq \mathbb{R}^{\mathbb{N}}$ be the set of "Fibonacci-type sequences," i.e., those sequences that satisfy the recurrence $a_n = a_{n-1} + a_{n-2}$ (for $n \geq 2$). Observe that F is closed under the left shift operator S. (You don't need to prove this.) (a) Show that F is a subspace: $F \leq \mathbb{R}^{\mathbb{N}}$. (b) Find the eigenvectors and eigenvalues of S in F. **(3+7 points)**

DO 15.32. IMPORTANT! Study the relevant chapters of the instructor's online "Discover Linear Algebra" text.