Discrete Math 37110 - Class 16 (2016-11-17)

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Partially revised by instructor. Errors possible. Consult online text and your notes.

Definition 16.1. An *abelian group* is a set a with a binary operation $A \times A \to A$ where we note $(a, b) \mapsto a + b$ which satisfies the following axioms:

(1) $(\forall a, b)(\exists! a + b)$

(2) $(\forall a, b, c)((a + b) + c = (a + (b + c))$

(3) $(\exists 0)((\forall a)(0 + a = a + 0 = a)$

(4) $(\forall a)(\exists b)(a + b = b + a = 0)$ (notated as $-a$)

(5) $(\forall a, b)(a + b = b + a)$ (this axiom is what makes a group abelian)

Example 16.2. Some examples of abelian groups:
$\mathbb{R}^n$, $(\mathbb{Z}, +)$, $\mathbb{R}^\Omega$, polynomials over $\mathbb{R}$

Muliplication by scalar: $\mathbb{R} \times A \to A$ where $(\lambda, a) \mapsto \lambda a$
Properties:

6. $(\forall \lambda \in \mathbb{R})(\forall a \in A)(\exists! \lambda a \in A)$
7. $(\forall \lambda, \mu \in \mathbb{R})(\forall a \in A)(\lambda(\mu a) = (\lambda\mu)a)$
8. $(\forall \lambda, \mu \in \mathbb{R})(\forall a \in A)((\lambda + \mu)a = \lambda a + \mu a)$
9. $(\forall \lambda \in \mathbb{R})(\forall a, b \in A)(\lambda(a + b) = \lambda a + \lambda b)$
10. $1 \cdot a = a$

DO 16.3. In a abelian group, 0 is unique and $-a$ is unique.

DO 16.4. $0 \cdot a = 0$ and $\lambda \cdot 0 = 0$

DO 16.5. If $\lambda \cdot a = 0$ then $\lambda = 0$ or $a = 0$

DO 16.6. Prove that $(-1)a = -a$

The empty list is linearly independent. $\{a\}$ is linearly independent if and only if $a \neq 0$.

DO 16.7. $\{a, b\}$ are linearly independent if and only if neither is a scalar multiple of the other

Notably, $[a, a]$ is linearly dependent.
If L_1 is a sublist of L_2 and L_2 is linearly independent, then L_1 is linearly independent.

Definition 16.8. Definition of polynomial degree: $\deg(f) = \max\{j \mid a_j \neq 0\}$
$\deg(0) = -\infty$

DO 16.9. $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$

DO 16.10. $\deg(f \cdot g) = \deg(f) + \deg(g)$

DO 16.11. Show that the above rules hold when f or g is 0

DO 16.12. $\deg(f) = 0$ if and only if f is a non-zero constant $f(t) = a_0$

DO 16.13. $f_0, f_1, \dots \in \mathbb{R}[t]$ if $\deg(f_i) = i$, then the polynomials are linearly independent.

DO 16.14. Prove: if $S \subseteq V$ then the set of all linear combinations of S is a subspace.

HW 16.15. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Compute A^n. Your answer should be a closed-form expression for each of the four entries, using familiar functions. **(5 points)**

HW 16.16. Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Compute B^n (closed-form). **(5 points)**

DO 16.17. A strictly upper triangular matrix $A \in M_n(\mathbb{R})$ is one where the diagonal and all values below the diagonal are 0. Then $A^n = 0$

DO 16.18. If A is nilpotent (some power of A is zero) then $\exists (I - A)^{-1}$