Discrete Math 37110 - Class 18 (2016-11-29)

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18.1 Random Walks/Finite Markov Chains

Check LN on this subject!

The set of states is Ω. Time is discrete: $t = 0, 1, 2 \ldots$ X_t describes the state at time t The transition probabilities are: $p_{ij} = P(X_{t+1} = j \mid X_t = i)$. Note they do not depend on t (the system has no memory).

Note that $p_{ij} \geq 0$ and $(\forall i)(\sum_{j=1}^{n} p_{ij} = 1)$.

Definition 18.1 (Transition matrix)**.** The MC is described by its transition matrix $T = (p_{ij}) \in M_n(\mathbb{R})$.

Definition 18.2. $A \in M_n(\mathbb{R})$ is a *stochastic matrix* if every row is a probability distribution, i.e., $(\forall ij)(a_{ij} \geq 0)$ and $(\forall i)(\sum_j a_{ij} = 1)$.

DO 18.3. Stochastic matrices are precisely the transition matrices of finite Markov Chains.

The distribution of the particle at time t is $q_t = (q_{t,1}, \ldots, q_{t,n})$ where $q_{t,i} = P(X_t = i)$. For each t, the vector q_t is a probability distribution.

DO 18.4 (Evolution of Markov Chains)**.** Show that $q_{t+1} = q_t \cdot T$

Corollary 18.5 (Evolution of Markov Chains)**.** $q_t = q_0 T^t$

DO 18.6. The t-step transition probability: $P(X_{s+t} = j \mid X_s = i) = p_{ij}^{(t)}$ Prove: $T^t = (p_{ij}^{(t)})$.

HW 18.7. Let $T = \begin{bmatrix} 0.3 & 0.7 \\ 0.2 & 0.8 \end{bmatrix}$. Prove: $\exists L = \lim_{t\to\infty} T^t$. Find the limit and prove. Prove $(\exists 0 < c < 1)(\forall t)\, |T^t - L| \leq c^t$ where $|A|$ is defined as $\max_{i,j} |a_{ij}|$. **(9 points)**

Definition 18.8. $q \in \mathbb{R}^n$ is a **stationary distribution** for our MC if q is a distribution and $qT = q$.

DO 18.9. Prove: if $\exists \lim_{t\to\infty} q_t$ then this limit is a stationary distribution.

DO 18.10. If $\exists L = \lim_{t\to\infty} T^t$ then every row of L is a stationary distribution.

DO 18.11. Prove: 1 is a right eigenvalue of every stochastic matrix.

Proof. Each row sums to 1 so the all-ones vector $\mathbf{1} = (1, 1, \ldots, 1)^T$ is a right eigenvector to eigenvalue 1. □

DO 18.12. For all $n \times n$ matrices, the right and left eigenvalues are the same.

Proof. $\det(I - tA) = \det(I - tA^T)$. □

Theorem 18.13 (Perron–Frobenius Theorem)**.** *Suppose $A \in M_n(\mathbb{R})$, $A = (a_{ij})$, $a_{ij} \geq 0$ (i.e., A is a non-negative matrix). Let us number the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then we can choose λ_1 to be real and non-negative. Moreover, λ_1 will have a non-negative eigenvector.*

DO 18.14. If T is a stochastic matrix, and $\lambda \in \mathbb{C}$ is an eigenvalue of T, then $|\lambda| \leq 1$.

Corollary 18.15. *Every MC has a stationary distribution.*

Proof. By Perron–Frobenius and the preceding exercise, $\exists$ left eigenvector $x = (x_1, \ldots, x_n) \geq 0$ to eigenvalue 1. Now $q = \frac{1}{\sum x_i} x$ is a stationary distribution. □

Definition 18.16 (Digraph associated with a matrix)**.** Let $A = (a_{ij})$ and $n \times n$ matrix. We associate with A the digraph $G(A)$ with vertex set $[n]$; we have the edge $i \to j$ if $a_{ij} \neq 0$.

Definition 18.17 (Transition digraph of MC)**.** The **transition digraph** associated with our Markov Chain is $G(T)$. In other words, the set of vertices is $[n]$ and we have $i \to j$ exactly if $p_{ij} > 0$.

HW 18.18. Find a finite Markov Chain with more than one stationary distribution. Minimize the number of states. Submit a drawing of the transition digraph. Label each edge with the corresponding transition probability. **(7 points)**

Definition 18.19. An $n \times n$ matrix A is **irreducible** if its associated digraph $G(A)$ is strongly connected. A Markov Chain is **irreducible** if T is irreducible, i.e., if $G(T)$ is strongly connected.

Definition 18.20. A strong component C of a digraph is a *terminal* strong component if there is no edge going out of C (it is impossible to leave C).

DO 18.21. Prove: every digraph has a terminal strong component.

DO 18.22. Prove that if a state is not in a terminal strong component, the stationary probability is 0 in any stationary distribution.

DO 18.23. If the Markov Chain is irreducible then the stationary distribution is unique.

DO 18.24. The stationary distribution is unique if and only if there is exactly one terminal strong component.

Definition 18.25. We define the **period** of a vertex x in a digraph to be the gcd of the lengths of all closed walks through x.

DO 18.26. If x, y are in the same strong component, then they have the same period.

DO 18.27. If G is strongly connected then the period of G is the gcd of the lengths of all cycles.

HW 18.28. Draw a strongly connected digraph of period 3 without a cycle of length 3 and with a minimal number of edges. **(5 points)**

DO 18.29. Let A be an $n \times n$ matrix. If $G(A)$ is strongly connected and the period of $G(A)$ is d and λ is an eigenvalue of A, then $\lambda\omega$ is also an eigenvalue for all d-th roots of unity ω (i.e. $\omega^d = 1$).

Theorem 18.30 (Perron–Frobenius, 2nd part)**.** *If a nonnegative matrix is irreducible then λ_1 is a simple eigenvalue (has multiplicity 1 in the characteristic polynomial) and the same holds for $\lambda_1\omega$ for each d-th root of unity ω.*

Definition 18.31. The digraph G is **aperiodic** if it is strongly connected and has a period of 1.

DO 18.32. Which undirected graphs are aperiodic? (View the undirected graph as a digraph with each pair $\{u, v\}$ of adjacent vertices having two edges, $u \to v$ and $v \to u$ between them.)

Definition 18.33. A finite Markov Chain is **ergodic** if its transition digraphs is strongly connected and aperiodic.

Theorem 18.34. *If a Markov Chain is ergodic then $\exists \lim_{t\to\infty} T^t$.*

DO 18.35. This limit has a rank of 1.