CMSC-37110 Discrete Mathematics
MIDTERM Tuesday, November 8, 2016

Name (print): ________________________________

This exam contributes 20% to your course grade. email:

Do **not** *use book, notes, scrap paper.* Use the space provided; you may continue on the **back of each sheet. Show all your work.** If you are not sure of the meaning of a problem, **ask the proctor.** The XC problems are underrated, work on them after having done your best on the non-XC problems.

READ THIS!

In all problems involving a graph G, we use the following notation: n denotes the number of vertices, m the number of edges, $\chi(G)$ the chromatic number, $\alpha(G)$ the independence number of G.

If a problem speaks about a random element of a given set and does not specify a distribution, the distribution is assumed to be uniform.

You may use results proved in class except if the question is essentially the same as the result proved in clas; in the latter case you are asked to reproduce the proof.

In addition, you may use the following facts.

- If G is a planar graph and $n \geq 3$ then $m \leq 3n - 6$.
- If G is a triangle-free planar graph and $n \geq 3$ then $m \leq 2n - 4$.
- (Prime Number Theorem) $\pi(x) \sim x/\ln x$ where $\pi(x)$ denotes the number of primes $\leq x$.

---

1. (4 points) Spelling: fill in the blanks.
   Use clearly legible BLOCK LETTERS (no cursive).

   If there is an edge between two VERT _ _ _ S, x and y, of a graph, we say x and y are ADJ _ CE _ _ .

   The ADJ _ CE _ _ _ relation is symmetric and irreflexive.

2. (10 points) Let a_n be a sequence of real numbers. Define the statement $\lim_{n\to\infty} a_n = \infty$. Your answer should be a well-quantified expression with no English words.

3. (14 points) Let $f(n)$ denote the number of binary digits of n and $g(n)$ the number of decimal digits of n. (So for instance $f(13) = 4$ because 13 in binary is 1101, and $g(13) = 2$.) True or false: $f(n) = \Theta(g(n))$. Prove your answer.

4. (11 points) Let $k \geq 1$. Find the multiplicative inverse of k modulo $k^2 + k + 1$. Your answer should be a number between 0 and $k^2 + k$.

5. (12 points) Recall that $\pi(n)$ denotes the number of primes $\leq n$. Decide which of the following three statements, involving the little-oh notation, is true. Circle and prove your answer.
(a) $\pi(n) = o(n^{0.9})$ (b) $n^{0.9} = o(\pi(n))$ (c) neither. Prove your answer.

6. (12 points) Suppose $a^{21} \equiv 1 \pmod{71}$. Prove: $a^7 \equiv 1 \pmod{71}$.

7. (3+4+5+15 points) Recall that an *orientation* of a graph G is a digraph obtained by assigning a direction to each edge of G. So the number of orientations of a graph is 2^m.
Let G be the $k \times \ell$ grid graph where $k, \ell \geq 2$. (So this graph has $n = k\ell$ vertices.)

   (a) Let N denote the number of 4-cycles in this graph. Determine N.

   (b) Let $\vec{G}$ denote a random orientation of G. What is the size of the sample space for this experiment?

   (c) Let X denote the number of directed 4-cycles in $\vec{G}$. Determine $E(X)$.

   (d) Determine $\mathrm{Var}(X)$.

   Your answers should be simple closed-form expressions in terms of k and ℓ. Prove your answers.

8. (13+8XC points)

   (a) Prove: the average degree of a planar graph is less than 6. (The average degree is the average of the degrees of all vertices.)

   (b) (XC) Prove: for every positive ϵ there exists a planar graph with average degree $\geq 6 - \epsilon$.

9. (16 points) Let us consider a random string w of length n over the 5-letter alphabet $\{A, B, C, D, E\}$. Determine the probability that all the five letters actually occur in w. Your answer should be a closed-form expression. (Hint: inclusion–exclusion.) State the size of the sample space.

10. (13+8XC points)

(a) Prove: if the graph G is triangle-free then $\chi(G) = O(\sqrt{n})$.

(b) (XC) Prove: if the graph G has no 4-cycles then $\chi(G) = O(\sqrt{n})$.

11. (20 points) Prove: for almost all graphs $G \sim \mathbb{G}_{n,1/2}$ we have $\alpha(G) \le 1 + 2\log_2 n$. Define what "almost all" means in this context.

12. (9 points) Prove the identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

13. (9 points) Find a system of 3 simultaneous congruences of the form $x \equiv a_i \pmod{m_i}$ $(i = 1, 2, 3)$ such that (a) no two of the m_i are relatively prime; (b) no two of the a_i are equal; and (c) the system has a solution. State the value M such that your system has a unique solution modulo M. No proofs necessary.

14. (7+14 points) (a) Count the strictly increasing functions $f : [k] \to [n]$. ("Strictly increasing" means $f(i+1) > f(i)$ for every i.) (Recall the notation $[k] = \{1, \ldots, k\}$.) (b) Count those functions $f : [k] \to [n]$ that satisfy $f(i+1) \geq f(i)+2$ for every i. Your answers should be simple expressions involving binomial coefficients. Prove your answers.

15. (9 points) Draw a triangle-free graph with 11 vertices and chromatic number 4. Your drawing should display a rotational symmetry of oder 5 (rotation by 72^o should not change the graph). No proof necessary.

16. (XC: 8 points) Prove: for all sufficiently large n, the number of non-isomorphic trees with n vertices is greater than 2.7^n. Use Cayley's formula n^{n-2}; state what it is that this formula counts. (Recall: $\mathrm{e} = 2.718\ldots$.)

17. (XC 10 points) Prove: $9 \to (4, 3)$.

18. (XC 15 points) Let $A_1, \ldots, A_m$ be events such that $(\forall i)(P(A_i) = 1/2)$ and $(\forall i \neq j)(P(A_i \cap A_j) \leq 1/5)$. Prove: $m \leq 6$.

19. (XC 12 points) Prove: If we pick $n+1$ elements from the set $\{1, \ldots, 2n\}$ then one of them will divide another one.

20. (XC 8 points) Let p be a prime number and let $f(x) = 1 + x + x^2 + \cdots + x^{p-2}$. Prove: $(\forall x)(f(x) \equiv -1, 0 \text{ or } 1 \pmod{p})$.

21. (XC 7 points) Let a and b be n-digit positive numbers. We apply Euclid's algorithm to determine $\gcd(a, b)$. Prove: the algorithm terminates in $O(n)$ rounds. (One round consists of one application of the Division Theorem.)

22. (XC 7 points) The *girth* of a graph is the length of its shortest cycle. Prove: if every vertex of a graph G has degree ≥ 3 then the girth of G is $O(\log n)$.