CMSC-37110 Discrete Mathematics

FINAL EXAM Tuesday, December 5, 2017, 10:30-12:30

Name (print): ____________________________

Email:

This exam contributes 40% to your course grade.

Do **not** *use book, notes, scrap paper.* NO ELECTRONIC DEVICES. Use the space provided on the exam sheets; you may continue on the **back of each sheet. Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The XC ("extra credit") problems are underrated, work on them after having done your best on the non-XC problems.

READ THIS!

You may use results stated in class or in exercises except if the question is essentially the same as the a result proved in class; in the latter case you are asked to reproduce the proof.

Here are some definitions and results from linear algebra for your convenience.

DEF. Two matrices $A, B \in M_n(\mathbb{R})$ are *similar* if there exists a nonsingular matrix $S \in M_n(\mathbb{R})$ such that $B = S^{-1}AS$.

DEF. A square matrix is *diagonalizable* if it is similar to a diagonal matrix.

DEF. An *eigenbasis* of a matrix $A \in M_n(\mathbb{R})$ is a basis of $\mathbb{R}^n$ such that every element of that basis is an eigenvector of A.

THM. A matrix $A \in M_n(\mathbb{R})$ is diagonalizable if and only if it has an eigenbasis.

A useful fact about *planar graphs*: if $n \geq 3$ then a planar graph with n vertices has at most $3n - 6$ edges.

1. (8+8+15 points)

(a) Does the equation $15x + 6y + 10 = 0$ have a solution in integers x, y ? Prove your answer.

(b) Does the equation $15x + 7y + 10 = 0$ have a solution in integers x, y ? Prove your answer.

(c) Determine all integer values of the number k such that the equation $15x + ky + 10 = 0$ has a solution in integers x, y. Your answer should be of the following form: such a solution exists if and only if $k \not\equiv a \pmod{m}$ for some small integers a and m. Determine a and m. Prove your answer.

Hint. Convert each of these questions into the solvability of a linear congruence. Or use the connection between gcd and linear combinations.

2. (12 points) Prove:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

3. (7 points) Find a sequence $a_1, a_2, \ldots$ of real numbers such that $\lim_{n\to\infty} a_n = \infty$ but infinitely often $a_n > a_{n+1}$.

4. (10 points) Construct a probability space (Ω, P) and two random variables, X and Y, that are uncorrelated but not independent. Make your sample space as small as possible. Prove that your variables work as required.

5. (8+6 points)

   (a) Define the concept of a *relation* on a set. Your answer should be a mathematically accurate English sentence that gives a complete definition. Begin your sentence with these words: "A relation on a set A is ...." As primitives you may use the concepts of sets, functions, and Cartesian products of sets.

   (b) If $|A| = n$, what is the number of relations on A? State, do not prove.

6. (8+7+15 points)

   (a) Let us perform n independent Bernoulli trials, each with probability p of success. Let $b(n, k, p)$ denote the probability that there will be exactly k successes among the n trials ($0 \leq k \leq n$). Give a simple closed-form expression for $b(n, k, p)$. (Your closed-form expression can use binomial coefficients.)

   (b) What is the expected number of successes? Show your work.

   (c) Prove that if $1 \leq k \leq np$ then $b(n, k-1, p) < b(n, k, p)$.

7. (14 points) Let $A \in M_n(\mathbb{R})$ be an $n \times n$ real matrix. Prove: the columns of A are linearly independent if and only if the columns of the matrix A^{100} are linearly independent.

8. (7+8+15 points) Let us fix a set V of n vertices.

   (a) Let $g(n)$ denote the number of graphs with V as their set of vertices. Find a simple closed-form expression for $g(n)$.

   (b) Prove: $\log_2 g(n) \sim an^b$ for some constants a, b. Determine a, b.

   (c) Let $t(n)$ denote the number of triangle-free graphs with V as their set of vertices. Prove: $\log_2 t(n) = \Omega(n^2)$. Recall: what this means is that there exists a constant $c > 0$ such that $\log_2 T(n) \geq cn^2$ for all sufficiently large n. State the largest c for which you can prove this.

   ("$\log_2$" denotes the base-2 logarithm.)

9. (8 points) Prove: every planar graph has a vertex of degree ≤ 5.

10. (14 points) Let $f(n)$ denote the number of binary digits of n and $g(n)$ the number of decimal digits of n. (So for instance $f(13) = 4$ because 13 in binary is 1101, and $g(13) = 2$.) True or false: $f(n) = \Theta(g(n))$. Prove your answer.

11. (15 points) For a non-negative integer n, let $s(n)$ denote the sum of the decimal digits of n. For instance, $s(0) = 0$ and $s(2017) = 10$. Pick a random number X uniformly from the set $\{0, 1, \ldots, 10^6 - 1\}$. Determine $E(s(X))$. Show your work.

12. (8+12 points)

(a) Find a real symmetric matrix A of which the characteristic polynomial is $f_A(t) = (t-1)(t+2)(t+5)^3$.

(b) Prove: if B is a real symmetric matrix then its characteristic polynomial $f_B(t)$ cannot be divisible by the polynomial $g(t) = t^2 + t + 1$.

13. (15 points) Find all pairs (x, y) of real numbers such that the following 3×3 matrix has rank 2. Your answer should be of the following form: a pair (x, y) makes the rank of this matrix equal to 2 if and only if x and y satisfy the linear equation $ax + by + c = 0$ for certain integers a, b, c where $\gcd(a, b, c) = 1$. Find a, b, c. Prove your answer. Hint. Cofactor expansion.

$$A = \begin{pmatrix} x & y & 5 \\ 1 & -1 & 7 \\ 2 & 0 & 3 \end{pmatrix}$$

14. (16 points) Consider the following four polynomials: $f_1(t) = t^2 + 27t + 2$, $f_2(t) = t^2 - 123$, $f_3(t) = 5t^2 - 87t + 166$, $f_4(t) = 94t - 10^6$. Decide whether these polynomials are linearly independent in the space $\mathbb{R}[t]$ of polynomials with real coefficients. Prove your answer. Give the name and the exact statement of the result from class you are using. This should be a central result. Hint. No calculation is needed.

15. (8 + 20 points) Let G be a k-regular graph with n vertices. (k is the degree of each vertex.) Let us color the vertices red and blue using n independent Bernoulli trials, each with probability p of success; if the i-th trial succeeds, the i-th vertex gets colored red. Next we delete all edges that have one or two blue endpoints. Let X denote the number of remaining edges, so these are the red-red edges.

    (a) Determine $E(X)$.

    (b) Determine $\text{Var}(X)$.

    Your answers should be simple closed-form expressions.

16. (7 + 18 points)

   (a) Find a 2×2 matrix $A \in M_2(\mathbb{R})$ such that the characteristic polynomial of A is $f_A(t) = t^2$ but $A \neq 0$.

   (b) Prove for every n: if a symmetric real matrix $A \in M_n(\mathbb{R})$ has characteristic polynomial $f_A(t) = t^n$ then $A = 0$.

17. (7 + 7 + 7 points) Let $n \geq 2$. Consider the following three statements. Each statement begins with the same phrase (next line).

   If $A \in M_n(\mathbb{R})$ is a symmetric real matrix, then

   (a) A is nonsingular TRUE FALSE

   (b) A has n distinct eigenvalues TRUE FALSE

   (c) A is diagonalizable TRUE FALSE

   For each statement, circle your anser (true or false). If true, prove for every n; if false, give a counterexample for $n = 3$. You may use the Spectral Theorem.

18. (18 + 8 points)

(a) Decide whether the following system of simultaneous congruences has a solution. Circle your answer: YES NO

$$\begin{aligned} x &\equiv 7 \pmod{24} \\ x &\equiv 43 \pmod{90} \\ x &\equiv 63 \pmod{100} \end{aligned}$$

Prove your answer. In case your answer is YES, you are not required to calculate a solution, just to prove that a solution exists.

(b) If a system of simultaneous congruences of the form

$$\begin{aligned} x &\equiv a \pmod{24} \\ x &\equiv b \pmod{90} \\ x &\equiv c \pmod{100} \end{aligned}$$

does have a solution, the solution is necessarily unique modulo some number m. Find m and say how you got it.

19. (6 + 12 + 20 points) Let θ be a real number. Prove that the matrix A_θ below is diagonalizable for all but one value of θ. (a) Find this exceptional value, call it α. (b) For all $\theta \neq \alpha$ prove that A_θ is diagonalizable and find the diagonal matrix similar to A_θ. (c) Prove that the matrix A_α is not diagonalizable. — Prove your answers. No calculation is needed for the proofs.

$$A_\theta = \begin{pmatrix} -5 & -17 & \sqrt{3} \\ 0 & \cos\theta & 9 \\ 0 & 0 & \mathrm{e}^{|\theta|} \end{pmatrix}$$

Note: on the test, this matrix had a typo (5 instead of -5). With that typo, there are three exceptional values of θ.

20. (4+7+15 points)

(a) Let us consider a random function $f : [k] \to [n]$ (chosen uniformly form among all such functions). What is the size of the sample space for this experiment?

(b) Let $p(n, k)$ denote the probability that f is injective, i. e., that there are no collisions: $f(x) = f(y) \implies x = y$. Give a simple closed-form expression for $p(n, k)$ (using factorials and possibly binomial coefficients).

(c) Prove: if $k = k_n = \omega(\sqrt{n})$ then $\lim_{n\to\infty} p(n, k_n) = 0$. (Recall that $f = \omega(g)$ means $g = o(f)$.)

21. (XC 10 points) Pick a random graph uniformly from all graphs on a given set of n vertices. Let p_n denote the probability that the graph is planar. Prove: for all sufficiently large n we have

$$p_n < 2^{-0.49n^2}$$

22. (XC 12 points) Let $T(n)$ denote the number of transitive relations on a given set of n elements. Prove: $\log_2 T(n) = \Theta(n^2)$. — In other words, you have to show that there exist positive constants c_1, c_2 such that $c_1 n^2 \leq \log_2 T(n) \leq c_2 n^2$ for all sufficiently large n. Specify your constants. The closer they are to each other, the better.

23. (XC 8 points) Prove: the chromatic number of a triangle-free graph on n vertices is $O(\sqrt{n})$.

24. (XC 12 points) Prove: almost all graphs have diameter 2. Explanation. The distance between two vertices is the length of a shortest path between them. The diameter of a graph is the maximum distance between pairs of vertices. Let p_n be the probability that a random graph on vertex set $[n]$ (picked uniformly) has diameter 2. You need to show that $\lim_{n\to\infty} p_n = 1$.

25. (XC 7 points) Let $A, B \in M_n(\mathbb{R})$. Prove: $AB - BA \neq I$.

26. (XC 10 points) Let G be a self-complementary graph with n vertices (i. e., G is isomorphic to its complement). Prove: $\chi(G) \geq \sqrt{n}$.

27. (XC 9 points) The *girth* of a graph is the length of its shortest cycle. For instance, the girth of the grid is 4, and the girth of the Petersen graph is 5.

    Prove: there exists a non-planar graph of girth ≥ 100.