Algorithms in Finite Groups
CMSC 36500 = Math 37500
Final test – May 30, 2017

Name (print): ________________________________

This quiz contributes 42.5% to your course grade. MIN: email:

Do **not** *use book, notes, scrap paper.* Use the space provided; you may continue on the **back of each sheet.** You can find some of the definitions on the last page.

Show all your work. If you are not sure of the meaning of a problem or the degree of detail required, **ask the instructor.**

The BONUS PROBLEMS are underrated, try the ordinary problems first.

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1. (18 points) True or false: If the group G is a semidirect product or two nilpotent subgroups then G is nilpotent. Prove your answer.

2. (28 points) Prove that the Johnson scheme $\mathfrak{J}(k,t)$ ($k \geq 2t+1$) is a primitive coherent configuration. Do not use groups in the proof; prove this directly from the definition of primitivity of a coherent configurations. You do not need to prove coherence, only primitivity assuming coherence.

3. (40 points) Prove: Graph Isomorphism is Karp-reducible to isomorphism of regular graphs.

4. (30 points) For infinitely many values of n construct a connected graph X with n vertices such that the vertices of X have only two different degrees, yet naive refinement completely splits X (each vertex receives a different color).

5. (22+22 points)

(a) Prove: If a primitive permutation group $G \leq S_n$ has a nontrivial solvable normal subgroup then n, the degree of G, is a prime power.

(b) Prove: for every prime power $n = p^k$ there exists a primitive group of degree n with a nontrivial solvable normal subgroup.

6. (20 points) Count the subgroups of order p^2 in the group $\mathbb{Z}_p^5$ (p is a prime). Your answer should be a closed-form expression (no dot-dot-dots, no product of summation signs).

7. (40 points) Let $G = T_1 \times \cdots \times T_k$ be a direct product of the nonabelian finite simple groups T_i. Prove that G has exactly 2^k normal subgroups.

8. (30 points) Let $p \geq 5$ be a prime and let $d \geq 2$. Name the unique non-abelian composition factor of $GL(d, p)$. Define this group as a quotient of a subgroup of $GL(d, p)$ (quotient of what subgroup by what normal subgroup?). You do not need to prove that the group you defined is simple.

9. (25+15 points)

   (a) Prove: If $G \leq S_n$ is transitive then G has at most $n-1$ maximal systems of imprimitivity (the blocks are minimal).

   (b) Prove that this bound is tight for infinitely many values of n.

10. (20 points) Let $\mathfrak{X} = (V, c)$ be a coherent configuration and R_i a homogeneous constituent of $\mathfrak{X}$. Let W be the set of vertices of a strong component of R_i. Is the restriction of $\mathfrak{X}$ to W necessarily a coherent configuration? Decide and prove.

11. (35 points) Prove: the symmetry defect of a nontrivial semiregular bipartite graph is $\geq 1/2$ in each part.

12. (40 points) Let $\mathfrak{X} = (V, c)$ be a coherent configuration and R_i a constituent of $\mathfrak{X}$. Let Y be a weak component of R_i. Suppose Y is a directed cycle of length k. Prove: every weak component of R_i is isomorphic to Y.

13. (40 points)[Twin awareness] Let $\mathfrak{X} = (V, c)$ be a coherent configuration and let $x, y, x', y' \in V$. Let R_i be a constituent of $\mathfrak{X}$ (consisting of the edges of color i). Assume x, y are twins with respect to R_i and $c(x, y) = c(x', y')$. Prove: x', y' are twins. — You may make the following additional assumptions: (a) R_i is bipartite; (b) $\mathfrak{X}$ has exactly two vertex-color classes, A and B. So $A \sqcup B = V$ and $R_i \subseteq A \times B$.

14. (BONUS: 25B points) Let X be a graph such that every eigenvalue of X has multiplicity ≤ 2. Prove: $\mathrm{Aut}(X)$ is solvable.

15. (BONUS: 15B+15B points)

(a) Let X, Y be graphs. Prove: If for every $k \geq 0$, the graph X has the same number of closed walks of length k as Y then X and Y have the same characteristic polynomial.

(b) Prove: If X and Y are graphs whose characteristic polynomials are not the same then the (classical) Weisfeiler–Leman refinement proves they are not isomorphic.

16. (BONUS: 20B points) Let $G \leq S_n$ be a transitive permutation group. Prove: the number of structure trees of G is $n^{O(\log n)}$. (Note: the big-Oh notation denotes an upper bound.)

17. (BONUS: 20B points) Given a transitive group $G \leq S_n$ (by a list of generators), find all maximal systems of imprimitivity in polynomial time. State a clear mathematical lemma that underlies your algorithm.

DEFINITIONS

By *eigenvalues* of a graph we mean the eigenvalues of its adjacency matrix.

$GL(d, q)$ denotes the group of invertible $d \times d$ matrices over the field $\mathbb{F}_q$ where q is a prime power.

Let X be a digraph. We say that two vertices u, v are **twins** if the transposition (u, v) is an automorphism of X.

By a *bipartite graph* we mean a triple $X = (V_1, V_2; E)$ where V_1 and V_2 are disjoint sets (the "parts") and $E \subseteq V_1 \times V_2$. We say that X is trivial if it is empty (i. e., $E = \emptyset$) or complete (i. e., $E = V_1 \times V_2$). We say that X is *semiregular* if each vertex in V_1 has the same out-degree and each vertex in V_2 has the same in-degree. Let T be a largest twin equivalence class in V_i. Then we say that the *symmetricity* of X in part i is $\alpha := |T|/|V_i|$ and the *symmetry defect* of X in part i is $1 - \alpha$.

Let $\mathfrak{X} = (V, c)$ be a coherent configuration where $c : V \times V \to \{\text{colors}\}$ is the edge-coloring. The *color-i constituent* of $\mathfrak{X}$ is the set $R_i = c^{-1}(i) \subseteq V \times V$. We know that R_i is either homogeneous (all of its vertices have the same color) or bipartite (of the form $R_i \subseteq C_1 \times C_2$ where C_1 and C_2 are two vertex-color classes).

Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive permutation group. A *structure tree* of G is a rooted tree of which Ω is the set of leaves such that the G-action extends the tree and the stabilizer of each internal (non-leaf) node x of the tree acts as a primitive group on the children of x.

The Johnson scheme $\mathfrak{J}(k, t)$ $(k \geq 2t + 1)$ has the vertex set $\binom{[k]}{t}$ (set of t-subsets of $[k]$); for $A, B \in \binom{[k]}{t}$ we set $c(A, B) = |A \setminus B|$ as the color of the pair (A, B).