Algorithms in Finite Groups
CMSC 36500 = Math 37500
Quiz – May 16, 2017

Name (print): ______________________________________

This quiz contributes 12.5% to your course grade.

Do **not** *use book, notes, scrap paper.* Use the space provided; you may continue on the **back of each sheet.** You can find some of the definitions on the last page.

Show all your work. If you are not sure of the meaning of a problem or the degree of detail required, **ask the instructor.**

The BONUS PROBLEMS are underrated, try the ordinary problems first.

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1. (7 points) Characterize the composition factors of finite solvable groups. (State the structure of the composition factors.) Do not prove.

2. (8 points) G acts on itself by conjugation. What is the kernel of this action? (Just name it, do not prove.)

3. (30 points) Prove: for $k \geq 2t+1$, the Johnson group $S_k^{(t)}$ is primitive.

4. (20 points) Prove: the Graph Isomorphism problem is Karp-reducible to the isomorphism problem for graphs of diameter 2. Your reduction should be very simple; simplicity counts. Describe the reduction and state what exactly we need to prove about your reduction; do not prove.

5. (30 points) Prove: Strongly regular graphs have ≤ 3 distinct eigenvalues.

6. (30 points) Let $N \triangleleft G$. Assume both N and G/N are nonabelian simple groups. Prove: N is a direct factor of G. In other words, there exists $L \leq G$ such that $G = N \times L$ (direct product). You may use a theorem stated but not proved in class. State the theorem.

7. (BONUS: 16 points) Let $n \geq 4$ be even. Prove:
$GL(n,2)$ has a subgroup isomorphic to the symmetric group S_{n+2}.
[Originally assigned for $n \geq 2$ but it is false for $n = 2$.]

8. (BONUS: 18 points) Prove: a primitive permutation group has at most two minimal normal subgroups.

DEFINITIONS

The permutation group G is *regular* if it is transitive and $|G_x| = 1$ for $x \in \Omega$.

Let $H \leq G$. The *centralizer* of H in G consists of those elements of G that commute with every element of G.

The Johnson group $S_k^{(t)}$ is defined as the induced action of S_k on the t-subsets of $[k]$. So

$$S_k \cong S_k^{(t)} \leq S_{\binom{k}{t}}.$$

The *distance* of vertices u and v in a graph is the length of the shortest $u - \ldots - v$ path. The *diameter* of a finite graph is the largest distance among pairs of vertices.

By *eigenvalues* of a graph we mean the eigenvalues of its adjacency matrix.

$GL(n,q)$ denotes the group of invertible $n \times n$ matrices over the field $\mathbb{F}_q$ where q is a prime power.

A graph is *strongly regular* (SR) with parameters (n,k,λ,μ) if it has n vertices, it is regular of degree k, evry pair of adjacent vertices has λ common neighbors, and every pair of non-adjacent vertices has μ common neighbors. For instance, the pentagon is SR with parameters $(5, 2, 0, 1)$.