Graph Isomorphism course, Spring 2017
Instructor: László Babai
Notes by Angela Wu, partially revised by instructor
Tuesday, May 9, 2017

13 Day 13, TuWk7

Homework from last time asked you to show that GI $\propto_{\text{KARP}}$ BIPARTITE GI.

Let $X \mapsto X'$ be the map on graphs that takes a graph $X = (V, E)$ and returns X', which is found by the addition of $|E|$ new nodes that bisect each edge of X.

DO 13.1. Show that $X \cong Y$ if and only if $X' \cong Y'$.

DO 13.2. If X is connected and trivalent and e is an edge in X, then $G = (\operatorname{Aut} X)_e$ is a 2-group.

Proof. Write $X = (V, E)$. Consider the Orbit-Stabilizer Theorem: $|G| = |x^G||G_x|$.

Let $T \subseteq V$. Let $G_{(T)}$ denote $(\operatorname{Aut} X)_{(T)}$, the pointwise stabilizer of T. Suppose that $v \notin T$ is a neighbor of a vertex in T. If $T' = T \cup \{v\}$, by orbit $v^{G_{(T)}}$ has length 1 or 2. So, $|G_{(T)} : G_{(T')}|$ is 1 or 2.

Let $T_0 = e = \{v_1, v_2\}$, so $G = G_{T_0}$. Let T_{i+1} be built from T_i by adding a neighbor v of T_i. By the above argument, $|G_{T_{i+1}} : G_{T_i}|$ is 1 or 2. Since G is a connected finite graph, $G_{(T_k)} = 1$ for some k. We have found that $|G|$ is a power of 2. □

13.1 Reduction of Trivalent GI to SI of 2-groups

We start with a set of vertices T. Denote by $N(T)$ the set of neighbors of T. We construct a layered graph: $L_0 = T$, $L_1 = N(T) \setminus T$, and $L_k = N(L_{k-1}) \setminus L_{k-1}$ for all k.

Two vertices in L_{k+1} are **twins** if their neighbors in L_k are identical.

Assume $G_{k-1} = \operatorname{Aut}(X_{k-1})$ is known. Suppose we want to compute G_k. Let $\varphi : G_k \to G_{k-1}$ be the restriction. This is a homomorphism because X_{k-1} is invariant under G_k. Notice that $\operatorname{Ker}(\varphi)$ is the direct product of symmetric groups on twin equivalence classes. We are interested in finding $\operatorname{Img}(\varphi) \leq G_{k-1}$.

DO 13.3. Show that $G_k = \langle \text{lifting of each generator of } \operatorname{Img}\varphi, \text{ generators of } \operatorname{Ker} \rangle$.

For each L_k, build the hypergraph on L_k given by the set of neighbors of vertices in L_{k+1}.

DO 13.4. If L_{k-1} has m vertices, then reduce finding $\operatorname{Img}(\varphi)$ to SI for strings of length $m + \binom{m}{2} + \binom{m}{3}$.

13.2 GI for bounded degree graphs is polynomial time

DO 13.5. Suppose that X is connected and d-valent. Show that $G = (\operatorname{Aut} X)_e$ is in Γ_{d-1}.

We give two definitions of Γ_d, a class of finite groups.

Definition 13.6. A group G is in Γ_d if there exists a subgroup chain of G

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = 1,$$

such that $|G_{i-1} : G_i| \leq d$.

Definition 13.7. A group G is in Γ_d if every composition factor embeds in S_d (isomorphic to a subgroup of S_d).

HW 13.8. Show that the two definitions of Γ_d are equivalent.

Notice that if $G \in \Gamma_d$, then $\Theta(G) \leq d$. The following Theorem follows as a corollary from the BCP-Theorem. It was shown previous to the BCP-Theorem by Luks via a different trick.

Theorem 13.9 (Luks (1980))**.** GI *for graphs of degree* $\leq d$ *is in polynomial time.*

13.3 Tournaments and Petersen Graph

A **tournament** is an orientation of the complete graph – a directed graph on V where, for all $u, v \in V$ with $u \neq v$, either $(u, v) \in E$ or $(v, u) \in E$.

HW 13.10. If X is a tournament, then $\mathrm{Aut}(X)$ is solvable.

Challenge 13.11 (B-Luks (1983))**.** Isomorphism of tournaments is decidable in $n^{O(\log n)}$.

HW 13.12. The automorphism group of the Petersen graph is isomorphic to S_5.

Central to the new GI algorithm is the "Unaffected stabilizers lemma." Look it up on arXiv.