Graph Isomorphism course, Spring 2017
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16 Day 16, ThWk8

16.1 Quiz questions

Discussed answers to # 1–4.

16.2 Socle of primitive permutation groups

Lemma 16.1. *The centralizer of a regular permutation group $H \le S_n$ in S_n is regular and is isomorphic to H.*

Proof. We can identify a regular group with its right regular representation. Suppose $H \curvearrowright \Omega$ is regular. Identify this H-action with H_R (the right regular action of H). From before, $C_{\mathrm{Sym}(H)}(H_R) = H_L \cong H$. □

DO 16.2. If $N_1, N_2 \lhd G$, then $[N_1, N_2] \le N_1 \cap N_2$.

Theorem 16.3. *If $G \le \mathrm{Sym}(\Omega)$ is primitive, then it has at most 2 minimal normal subgroups; if there are two, both are regular and they are isomorphic.*

Proof. Suppose $M_1 \ne M_2$ are minimal normal subgroups of G. Then, $M_1 \cap M_2 = 1$ (by minimality) and therefore $[M_1, M_2] = 1$ (by exercise 16.2). Since M_1 centralizes M_2 and M_2 is transitive, M_1 is semiregular. Since M_1 is also transitive, M_1 is regular. By the same argument, so is M_2.

By Lemma 16.1 we conclude that $M_2 = C_{\mathrm{Sym}(\Omega)}(M_1)$ and $M_1 \cong M_2$. If there was a third minimal normal subgroup $M_3 \ne M_1$, then by the same argument $M_3 = C_{\mathrm{Sym}(\Omega)}(M_1)$, so $M_3 = M_2$, a contradiction. □

Corollary 16.4. *If G is primitive, then $\mathrm{Soc}(G)$ is characteristically simple. So $\mathrm{Soc}(G) = T^k$ for some simple group T and some $k \ge 1$.*

Proof. Let M_1 be a minimal normal subgroup of G. Then M_1 is characteristically simple, so $M_1 \cong T^k$ for some simple group G. If $M_1 = \mathrm{Soc}(G)$, we are done. Alternatively, $\mathrm{Soc}(G) = M_1 \times M_2$ where $M_2 \cong M_1$ so $\mathrm{Soc}(G) \cong T^{2k}$. □

Corollary 16.5. *If G is primitive and has 2 minimal normal subgroups, then $|G| \le n^{1+\log_2 n}$.*

Proof. In this case M is regular. □

Corollary 16.6. *If $G \le S_n$ is primitive and $|G| > n^{1+\log_2 n}$, then $\mathrm{Soc}(G)$ is nonabelian and the unique minimal normal subgroup of G.*

Theorem 16.7 (Cameron (1981) and Maróti, CFSG)**.** *Let $n \ge 25$. If a primitive permutation group $G \le S_n$ satisfies $|G| > n^{1+\log_2 N}$, then $n = \binom{m}{t}^\ell$ and $\mathrm{Soc}(G) \cong (A_m)^\ell$, and*

$$(A_m^{(t)})^\ell \le G \le A_m^{(t)} \wr S_\ell. \tag{1}$$

16.3 Continuing Luks's approach

We are in the situation where the ambient group G is transitive. Let $\mathcal{B}$ be a minimal system of imprimitivity (blocks are maximal). Write $b = |\mathcal{B}|$. We have

$$\psi : G \twoheadrightarrow \widetilde{G} \leq \mathrm{Sym}(\mathcal{B}) = S_b.$$

This action is primitive, so $\widetilde{G}$ is a primitive group.

Suppose that $\widetilde{G}$ is large: $b \geq 25$ and $|\widetilde{G}| > b^{1+\log_2 b}$. We apply Cameron's theorem above to $\widetilde{G}$. So we have $b = \binom{m}{t}^{\ell}$ and $\widetilde{G}$ satisfies Equation (1). This equation yields a homomorphism $\widetilde{\varphi} : \widetilde{G} \to S_\ell$. Let now $\varphi : G \to S_\ell$ denote the composition of ψ and $\widetilde{\varphi}$; so this is a transitive G-action on $[\ell]$. Let $K = \mathrm{Ker}(\varphi)$.

▶ **Easy case** $\ell \geq 2$

In this case, our recipe is to DESCEND to $\mathrm{Ker}\,\varphi$.

DO 16.8. In this case, $|G : \mathrm{Ker}\,\varphi| \leq b$.

We need to compensate this multiplicative cost by significant progress. Progress will be measured by a dramatic reduction of the parameter b.

DO 16.9. (a) If $m \geq 5$ and $m \neq 6$ then $\mathrm{Aut}(S_m) = S_m$, i.e., $\mathrm{Out}(S_m) = 1$, but $|\mathrm{Out}(S_6)| = 2$.
(b) For $m \geq 5$ and $m \neq 6$, $\mathrm{Aut}(A_m) = S_m$. For $m = 6$, $|\mathrm{Aut}(A_m) : S_m| = 2$.

Use the preceding exercise to prove the following.

DO 16.10. If $m \geq 7$, then $(A_m^{(t)})^\ell \leq \mathrm{Ker}\,\varphi \leq (S_m^{(t)})^\ell$.

DO 16.11. The group $\mathrm{Ker}\,\varphi$ has a system of imprimitivity consisting of $\binom{m}{t}$ blocks.

Now $b = \binom{m}{t}^\ell$ so the number of blocks of $\mathrm{Ker}\,\varphi$ is $\binom{m}{t} \leq \sqrt{b}$, indeed a dramatic reduction (cannot be repeated more than $\log\log n$ times).

▶ **Hard case** $\ell = 1$

We have a Johnson group. This is where Luks's approach breaks down: there is no obvious way to reduce a meaningful parameter.

In this case, $b = \binom{m}{t}$. The action is $S_m^{(t)} \cong S_m$ or $A_m^{(t)} \cong A_m$. Let Γ be the domain with $|\Gamma| = m$.

$$G \twoheadrightarrow \widetilde{G} \cong \mathrm{giant}(m) \curvearrowright \Gamma,$$

where the action on Γ is as a giant.

Our job is to find a subgroup $M \leq \mathrm{Sym}(\Gamma)$ of exponential (in m) index such that $\varphi(\mathrm{Aut}(x)) \leq M$ where x is our input string.

As an intermediate step, we shall find a canonical structure, such as a graph, on Γ. We then use this structure to find one of the following, at quasipolynomial multiplicative cost.

- Canonical coloring with no dominant color
- Canonical equipartition of the dominant color class
- Canonical Johnson graph on the dominant color class

16.4 Symmetry defect

Let $X = (V, E)$ be a graph. We say that two vertices u, v are **twins** if the transposition (u, v) is an automorphism of X.

DO 16.12. The "twin or equal" relation is an equivalence relation on V.

Let T be a largest equivalence class of this equivalence relation. We call the proportion $|T|/|V|$ the **symmetricity** of X and $1 - |T|/|V|$ the **symmetry defect** of X.

DO 16.13. (a) The symmetry defect of a graph and its complement are the same.

(b) The symmetry defect of the complete graph is 0.

(c) The symmetry defect of the complete bipartite graph $K_{r,s}$ is $\min(r/n, s/n)$ where $n = r + s$.

(d) The symmetry defect of the disjoint union of the complete graphs $K_{n_1}, \ldots, K_{n_k}$ is $1 - n_j/n$ where $n_j = \max(n_1, \ldots, n_k)$ and $n = \sum_i n_i$.

HW 16.14. (a) Prove: the symmetry defect of a nontrivial regular graph is $\geq 1/2$. (In other words, the symmetricity of a nontrivial regular graph is $\leq 1/2$.) (b) Prove that this bound is tight for all even values of n, the number of vertices.

(A graph is *nontrivial* if it is not empty or complete.)
Check website for assigned reading, and additional homework problems.