Graph Isomorphism course, Spring 2017
Instructor: László Babai
Notes by Angela Wu and the instructor
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17 Day 17, TuWk9

17.1 Symmetry defect of regular graphs

Proposition 17.1. *If X is a nontrivial regular graph, then its symmetry defect is $\geq 1/2$.*

Recall that we say $u, v \in V$ are **twins** if the transposition $(u, v) \in \mathrm{Aut}(X)$. Also, $\mathrm{defect}(X) = 1 - |T|/n$, where T is the largest equivalence class of $V(X)$ under the equivalence relation of "twin or equal."

Notice that $\mathrm{defect}(K_n) = 0$.

DO 17.2. $\mathrm{defect}(X) = \mathrm{defect}(\bar{X})$, because $\mathrm{Aut}(X) = \mathrm{Aut}(\bar{X})$.

Lemma 17.3. *If X is k-regular and nonempty ($k \geq 1$), then all independent sets have size $\leq n/2$.*

Lemma 17.4. *If T is a set of twins in a graph X, then the induced subgraph $X[T]$ is empty or complete (T is either independent or a clique).*

Indeed, $\mathrm{Aut}(X[T]) = \mathrm{Sym}(T)$.

Proof of Prop. 17.1. Let T be a twin equivalence class. We need to show that $|T| \leq n/2$. Indeed, if T is an independent set then $|T| \leq n/2$ by Lemma 17.3. If T is a clique then T is an independent set in the complement of X and therefore again $|T| \leq n/2$. $\square$

Note: this proof is simpler than the one given in class.

For the remaining exercises, read the approrpriate sections of the instructor's paper (accessible from the course home page). Below we refer to that paper as [GIQ]. (Why?)

17.2 Naive refinement

One round of the naive vertex refinement process takes as input a vertex-colored graph $X = (V, E, f)$ where $f : V \to \{\text{colors}\}$ is the coloring, and outputs a refined coloring g defined as follows:

$g(v) = (f(v); d_i(v) \mid i \in \{colors\})$ where $d_i(v)$ denotes the number of neighbors of v of color i. The naive refinement process iterated this refinement step until the color partition stabilizes (no number of colors does not increase).

DO 17.5. A coloring is stable under naive refinement of and only if the color classes form an equitable partition.

(See [GIQ], Sec. 2.4.2 for the definition of equitable partitions.)

DO 17.6 (Abe Mowshowitz $\approx$ 1970)**.** If the characteristic polynomial of the adjacency matrix of a graph is irreducible over the rationals then naive refinement completely splits the graph.

Hint: show that under this assumption, only the discrete partition is equitable.

17.3 Coherent configurations (CCs)

Read [GIQ], Section 3 for the basics of the combinatorial theory of CCs. (CCs are called "classical CCs" in the paper.) Let $G \leq \mathrm{Sym}(V)$. Denote by $\mathfrak{X}(G)$ the orbital configuration of G (given by the orbits of G on $V \times V$).

DO 17.7. Show that the orbitals of a group G form a coherent configuration.

DO 17.8. $\mathfrak{X}(G)$ is homogeneous if and only if G is transitive.

HW 17.9. $\mathfrak{X}(G)$ is primitive if and only if G is primitive.

DO 17.10. Show that the Johnson scheme $\mathfrak{J}(k,t)$ is a primitive coherent configuration. — Give two proofs: one combinatorial (directly from the definition), the other using the fact that the group is $S_k^{(t)}$ is primitive. (Hint for the second proof: the Johnson schemes are Schurian.)

DO 17.11. $\mathrm{Aut}(\mathfrak{X}(G)) \geq G$.

HW 17.12. (Typo fixed and additional explanation added at 8pm on 5-20.) Find an infinite family of groups $G \leq S_n$ (for infinitely many values of n) such that $|G|$ is small (grows slowly as a function of n) but $|\mathrm{Aut}(\mathfrak{X}G)|$ is very large (grows fast as a function on n). State conceptually what "small" and "very large" mean in terms of commonly used rates of growth (as functions of n).

HW 17.13. Prove that $c(x,y)$ is aware of $c(x)$ and $c(y)$.

DO 17.14. Each constituent of a CC is either biregular digraph or a semiregular bipartite graph.

The *directed distance* from vertex u to vertex v in a digraph is the length of the shortest directed path from u to v. The *directed diameter* of a digraph is the largest directed distance among pairs of vertices. (This quantity is infinite if the graph is not strongly connected.)

HW 17.15 (Due Tuesday 5/30)**.** All connected components of a homogeneous constituent of a CC have the same directed diameter.

(They also have the same undirected diameter, but you don't need to prove this.)

DO 17.16 (Useful for earlier exercises)**.** Let us fix a sequence $c_1, \ldots, c_t$ of (not necessarily distinct) colors. The number of walks of length t of color sequence $(c_1, \ldots, c_t)$ from x to y is determined by $c(x,y)$ and does not depend on the specific choice of x and y.