Graph Isomorphism course, Spring 2017
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This lecture was given by Bohdan Kivva
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HW due Tuesday, April 18

6 Day 6, ThWk3

Discussion of paper (continued)
L. Babai: On the Automorphism groups of strongly regular graphs (2014)
Click here for the paper

Theorem 6.1 (Main)**.** *Let X be a strongly regular graph that is not trivial or graphic. Then,*

$$\theta(\mathrm{Aut}(X)) = O\left(\frac{\ln^2(n)}{\ln\ln(n)}\right).$$

Recall that last time we proved the main theorem in the case that $k \geq n/4$. Today we will prove the main theorem for the case that $k < n/4$.

Definition 6.2. Let X be a k-regular graph and $k = \eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ be its eigenvalues. Then **zero-weight spectral radius for** X is the number $\eta = \max\{|\eta_i| : i = 2, \ldots, n\}$.

Lemma 6.3 (Expander Mixing Lemma)**.** *Let X be a k-regular graph with zero-weight spectral radius η. Let $S, T \subseteq V$. Then,*

$$\left| |E(S,T)| - \frac{|S|\,|T|\,k}{n} \right| \leq \eta\sqrt{|S|\,|T|}, \tag{1}$$

where $E(S,T) = E \cap S \times T$.

Proof. For a subset $W \subset V$, consider the indicator function $\chi_W : V \to \{0,1\}$. Let A be the adjacency matrix for X. Then, $|E(S,T)| = \chi_S^\top A \chi_T = \sum_{i\in S}\sum_{j\in T} a_{ij}$ while $|S|\,|T| = \chi_S^\top J \chi_T$, where J denotes the $n \times n$ all 1's matrix. It suffices to show that

$$\left| \chi_S^\top \left(A - \frac{k}{n}J\right)\chi_T \right| \leq \eta\sqrt{|S|\,|T|}. \tag{2}$$

Consider the eigenvalues of $A - \frac{k}{n}J$. The all 1's vector is an eigenvector with eigenvalue 0. The other eigenvalues are exactly $\eta_2, \ldots, \eta_n$. (Their corresponding eigenvectors are orthogonal to the all 1's vector.) □

Corollary 6.4. *Let X be a k-regular graph with zero-weight spectral radus η. Let $S \subset V$. Denote by $d(S)$ the average degree of vertices in the graph induced by S. Then, $\left|d(S) - \frac{k}{n}|S|\right| \leq \eta$.*

Proposition 6.5. *Le X be a k-regular graph with zero-weight spectral radius η. Suppose that any two distinct vertices in X has at most q common neighbors. Then, any nontrivial automorphism of X fixes at most $\frac{q+\eta}{k}n$ vertices.*

Proof. Let F be the set of fixed vertices of $\sigma \in \mathrm{Aut}(X)$. The average degree in the graph induced by $X \setminus F$ is at most $\frac{|X \setminus F|k}{n} + \eta$.

So, $q \geq k - \frac{|X \setminus F|k}{n} - \eta = k - \frac{(n-|F|)k}{n} - \eta = \frac{|F|k}{n} - \eta$. This is true if and only if $\frac{n(q+\eta)}{k} \geq |F|$. □

Proposition 6.6. *Let X be a nontrivial strongly regular graph. Then*

1. $\mu(n-k-1) = k(k-\lambda-1)$.
2. *X has three different eigenvalues $k > r > -s$ where $r \geq 1$ and $s \geq 1$.*
3. *If X is not a conference graph, i.e., the graph with parameters $\left(n, \frac{n-1}{2}, \frac{n-1}{4} - 1, \frac{n-1}{4}\right)$, then its eigenvalues are integers. If X is conference graph, then $r = \frac{-1+\sqrt{n}}{2}$ and $-s = \frac{-1-\sqrt{n}}{2}$.*
4. $r - s = \lambda - \mu$ *and* $rs = k - \mu$.

Proof. 1. Define $\Gamma_i(x)$ to be the set of vertices on distance i from x in X. Count $E(\Gamma_1(x), \Gamma_2(x))$ in two different ways.

2. Count paths of length 2 on X to find that $A^2 = (\lambda - \mu)A + (k-\mu)I + \mu J$, where J is the all 1's matrix. Then, an eigenvalue x of A satisfies $x^2 = (\lambda-\mu)x + (k-\mu)$, so there are at most two eigenvalues different from k. First, notice that if $\mu = k$, X is trivial (do exercise below). If $\mu < k$, we have one positive and one negative root. Denote them by r and $-s$. Note that X is connected, so $r < k$.

   We consider the multiplicities f and g of the eigenvalues r and $-s$ (HW below). If $f = 0$ or $g = 0$, let $\beta = r$ if $g = 0$ and $\beta = -s$ if $f = 0$. Then $A = \beta I + (k-\beta)J$, so X is trivial. Equation is true since the action of LHS and RHS on basis consisting of eigenvectors of A is the same. Thus we have exactly three different eigenvalues.

4. Follows from the quadratic equation for $r, -s$.

3. If $f \neq g$, then $-(f-g)(r+s) = \frac{1}{2}((r-s)(n-1)+2k) \in \mathbb{Q}$. So, $r+s$ is rational, so r and s are rational, so r and s are integers as roots of monic polynomial with integer coefficients.

   $f = g$ means that $(r-s)(n-1) + 2k = 0$. So, using part 4, $k = \frac{n-1}{2}$ and $\lambda = \mu - 1$, finally part 1 gives $\mu = \frac{n-1}{4}$. Moreover, solving quadratic equation in this case gives us $r = \frac{-1+\sqrt{n}}{2}$ and $-s = \frac{-1-\sqrt{n}}{2}$.

2. (again) Part 3 gives $r \geq 1$ and $s \geq 1$.

□

DO 6.7. If $\mu = k$, then X is the trivial graph.

HW 6.8. Prove that the multiplicities of the eigenvalues r and $-s$ are $f = \frac{1}{2}\left(n - 1 - \frac{(r-s)(n-1)+2k}{r+s}\right)$ and $g = \frac{1}{2}\left(n - 1 + \frac{(r-s)(n-1)+2k}{r+s}\right)$. Hint: consider the trace of A.

Lemma 6.9. *If X is a nontrivial strongly regular graph, with $s \geq 3$ and $k \leq n/4$ and $n \geq 25$, then*

$$\max(r,s) + \max(\lambda,\mu) \leq \frac{7}{8}k.$$

Proof. Case 1: Assume $\lambda \geq \mu$, then $r > s$. Then, $rs - r + s = k - \lambda$, so $(s-1)r + \lambda < k$, so $(s-1)(\lambda + r) < (s-2)\lambda + k < \left(\frac{3(s-2)}{4} + 1\right) k < \frac{7(s-1)}{8} k$.

Case 2: $\mu > \lambda$, then $s > r$. Since $3\mu n/4 \leq \mu(n-k) = k(k-\lambda-1) + \mu \leq k(k-\lambda) \leq k^2 \leq kn/4$, since $k \leq n/4$. So, $\mu \leq k/3$. If $r \geq 2$, then $(k-\mu)/2 \geq (k-\mu)/r = s$, so $s + \mu \leq (k+\mu)/2 \leq k/2 + k/6 \leq 4k/6 < 7k/8$. Observe, that if X is conference graph with $n \geq 25$, then $r \geq 2$. So the only other possibility is $r = 1$. If $r = 1$, then $s = \mu - \lambda + r \leq \mu - \lambda + 1 \leq \mu + 1$, so $s + \mu \leq 2k/3 + 1 \leq 7k/8$, which holds if and only if $16k + 24 \leq 21k$, which holds if and only if $5k \geq 24$, so $k \geq 5$.

Recall, $k^2 > n \cdot \min(\lambda, \mu)$, so $n < k^2$. □

Theorem 6.10 (Siedel)**.** *If X is a nontrivial strongly regular graph with $n \geq 29$. Suppose that $-s = -2$, then X is graphic ($X = L(K_n)$ or $L(K_{n,n})$ or the complement)*

Theorem 6.11. *If X is a nontrivial and nongraphic strongly regular graph with $n \geq 29$, then:*

1. *The order of any $G \leq \mathrm{Aut}(X)$ has order $\leq n^8$.*

2. *σ has at most $7n/8$ fixed points.*