TERSE NOTES GRAPH THEORY

AMIN IDELHAJ

1. 04/02/19

Remark 1.1. Singular of vertices is *not* vertice, but *vertex*

Definition 1.2. A graph is a pair $G = (V, E)$ where V is a set of vertices and E is a set of edges (ordered pairs of vertices).

Definition 1.3. Two vertices $u, v \in V$ are *adjacent* if $\{u, v\} \in E$, and we write $u \sim v$.

The adjacency relation is irreflexive, i.e. $(\forall\ u \in V)(u \not\sim u)$, and symmetric, i.e. $(\forall\ u, v \in V)(u \sim v \iff v \sim u)$. If we consider more than one graph on the same set of vertices, we write $u \sim_G v$.

Notation 1.4. If V is a set, then write $\binom{V}{2}$ for the set of unordered pair of elements of V.

Definition 1.5. If $G = (V, E)$ is a graph, then the complement is $\overline{G} = (V, \overline{E})$ where $\overline{E} = \binom{V}{2} \setminus E$

Given two distinct vertices u and v, we have that $u \sim_{\overline{G}} v \iff u \not\sim_G v$.

Definition 1.6. $[n] = \{1, \ldots, n\}$.

Notation 1.7. Given a graph $G = (V, E)$, we write $n = |V|$ for the *order* of the graph, and $m = |E|$ for the *size* of the graph. These letters are fixed.

Example 1.8. K_n is the complete graph on n vertices: $([n], \binom{[n]}{2})$.

For any graph, $0 \leq m \leq \binom{n}{2}$, where equality is achieved only by K_n. We refer to complete graphs as cliques.

Example 1.9. $\overline{K_n}$ is the empty graph.

Example 1.10. P_n is the path of length $n - 1$.

Remark 1.11. On general principle, the subscript of a graph name denotes the number of vertices.

Example 1.12. For $n \geq 3$, we have C_n, the cycle of length n. Here $m = n$.

Example 1.13. We have the $k \times l$ grid, where

DO 1.14. How many edges does the $k \times l$ grid have?

Example 1.15. $K_{r,s}$ is called the complete bipartite graph, where we take a vertex set of size r and of size s, and connect each vertex to all the vertices of the other set. Here, $n = r + s$ and $m = rs$.

Example 1.16. Q_d is called the d-dimensional cube. The vertex set is $\{0, 1\}^d$, and two strings are adjacent if they agree in all but one coordinate. $n = 2^d$.

Notation 1.17. If A is a set, then $|A|$ is the number of elements of A.

Definition 1.18. If $u \sim v$, then u and v are neighbors. Write $N_G(u)$ for the set of neighbors of u, and the *degree* of u is $\deg(u) = |N_G(u)|$.

Theorem 1.19 (Handshake)**.**

$$\sum_{v \in V} \deg(v) = 2m$$

DO! 1.20. Prove the Handshake theorem and figure out why it's called the Handshake theorem.

DO 1.21. Review relations, and in particular equivalence relations.

Definition 1.22. A graph G is regular of degree r if $(\forall\, v)(\deg(v) = r)$. We sometimes call it r-regular.

Examples 1.23.
- K_n is regular of degree $n-1$
- $\overline{K_n}$ is regular of degree 0
- C_n is regular of degree 2
- P_n is irregular except when $n = 1$, in which case it has degree 0, or $n = 2$, in which case it has degree 1
- grid(k, l) is irregular except when $k, l \leq 2$.
- $K_{r,s}$ is regular of degree r if and only if $r = s$.
- Q_d is regular of degree d

Definition 1.24. If $G = (V, E)$ and $H = (W, F)$ are graphs, then $f : V \to W$ is an *isomorphism* from G to H if it's bijective and $(\forall\, u, v \in V)(u \sim_G v \iff f(u) \sim_H f(v))$. If such an isomorphism exists, we say G and H are *isomorphic*, denoted $G \cong H$.

Example 1.25. Petersen's graph (I can't draw it quickly enough but Google Images). $n = 10$, it's 3-regular.

Remark 1.26. The previous example did not have a typo, it's spelled Petersen and not Peterson.

DO 1.27. The graph drawn on the board is isomorphic to the Petersen graph.

Definition 1.28. A graph is self-complementary if $G \cong \overline{G}$.

Examples 1.29. Some examples of self-complementary graphs include P_1, P_4, and C_5

HW 1.30 (Due Thursday)**.** If G is self-complementary, then $n \equiv 0$ or $1 \bmod 4$.

Definition 1.31 (SUBGRAPH)**.** If $G = (V, E)$ and $H = (W, F)$ are graphs, then H is a subgraph of G, denoted $H \subset G$, if $W \subset V$ and $F \subset E$.

Examples 1.32. $C_5 \subset K_5$ (no vertices deleted), $K_4 \subset K_5$ (a vertex deleted).

Examples 1.33. There are $\binom{n}{3}$ 3-cycles in K_n, since we just choose 3 vertices. Now, we can find 3 copies of C_4 in K_4, so there are $3\binom{n}{4}$ copies in K_n.

HW 1.34. Count the C_4 subgraphs of Q_d.

Theorem 1.35 (Mantel-Turan)**.** *If G is triangle free, i.e. $K_3 \not\subset G$, then $m \leq \frac{n^2}{4}$.*

DO* 1.36. Prove Mantel-Turan.

DO 1.37. The bound in Mantel-Turan obviously implies $m \leq \lfloor \frac{n^2}{4} \rfloor$, so show that this is tight (for every n, find an example achieving equality).

Definition 1.38. For $u, v \in V$, we say v is *accessible* from u if there exists a path starting at u and ending at v, written $u \dots v$

DO 1.39. Accessibility is an equivalence relation.

Remark 1.40. It's obviously reflexive and symmetric, but your immediate instinct for transitivity won't work right away, think about it carefully.

Definition 1.41. We refer to equivalence classes of the accessibility relation as *connected components*, and $k(G)$ is the number of them. G is *connected* if $k(G) = 1$.

DO 1.42. If G is connected then $m \geq n - 1$.

HW 1.43 (Due Thursday)**.** Given n, find the largest m such there exists a disconnected graph G with n vertices and m. edges.

Definition 1.44. A *tree* is a connected, cycle-free graph.

DO 1.45. If T is a tree then $m = n - 1$

HW 1.46 (Due Thursday)**.** Draw all the 7-vertex trees up to isomorphism and state how many there are.