NOTES GRAPH THEORY 04-04

AMIN IDELHAJ

Fibonacci Numbers

We have the usual Fibonacci numbers, defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, $F_0 = 0$ and $F_1 = 1$. So $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$, and so on.

Definition 0.1. Given some sequence $\underline{a} = (a_0, a_1, a_2, \ldots)$, we say that $\underline{a}$ is of Fibonacci type if $a_n = a_{n-1} + a_{n-2}$.

DO 0.2. The geometric progression $(1, q, q^2, \ldots)$ is of Fibonacci type $\iff q = \frac{1 \pm \sqrt{5}}{2}$

DO 0.3. Every Fibonacci-type sequence can be written as a linear combination of $(1, q_1, q_1^2, \ldots)$ and $(1, q_2, q_2^2, \ldots)$

DO 0.4. Find α and β such that $F_n = \alpha q_1^n + \beta q_2^n$.

Corollary 0.5. *The Fibonacci numbers have the form:*

$$F_n = \left\lfloor \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right\rceil$$

where the brackets denote the nearest integer.

Mantel-Turán Theorem

Theorem 0.6. *If G is a triangle-free graph, then $m \leq \frac{n^2}{4}$.*

Definition 0.7. If a_n and b_n are sequences, then a_n is *asymptotically equal* to b_n, denoted $a_n \sim b_n$, if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

In particular, note that for all graphs we have a bound of $m \leq \binom{n}{2} = \frac{n(n-1)}{2} \sim \frac{n^2}{2}$. So triangle-free graphs have roughly half the edges by Mantel-Turán. To prove this, we begin with a lemma.

Lemma 0.8. *If $x, y \in V$ and $x \sim y$, then* $\deg(x) + \deg(y) \leq n$

Proof. Given $u \in V$, then u is adjacent to at most one of x and y, otherwise we'd have a triangle $u \to x \to y \to u$. □

Now we give two proofs of Mantel-Turán. The first is inductive:

Proof. If $m = 0$ we're done. Otherwise pick an edge $\{x, y\}$ and let $G' = G - x - y$. Then we lose the edge $\{x, y\}$ and at most $n - 2$ other edges by the lemma, so $m_G \leq 1 + (n-2) + \frac{(}{n-2)}4 \leq \frac{n^2}{4}$. Now $n = 1$ and $n - 2$ are obvious, so the induction is complete. □

Before giving the second proof, we give some more definitions.

Definitions 0.9. Given $x_1, \ldots, x_n \in \mathbb{R}$, we define their *arithmetic mean* to be $A(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n}$, and their *quadratic mean* to be $Q(x_1, \ldots, x_n) = \sqrt{\frac{x_1^2 + \ldots + x_n^2}{n}}$.

DO 0.10. $A \leq Q$, with equality if and only if $x_1 = \ldots = x_n$.

Proof. The lemma gives us m inequalities, one for each edge. If we add them up all the terms on the LHS, then $\deg(x)$ appears $\deg(x)$-many times, once for each edge connecting to x, so:

$$\sum_{\{x,y\}\in E} \deg(x) + \deg(y) = \sum_{x\in V} \deg(x)^2$$

Then by the lemma, the DO exercise, and the Handshake theorem, we have:

$$\frac{(2m)^2}{n} = \frac{(\sum_{x\in V} \deg(x))^2}{n} \leq \sum_{x\in V} (\deg(x))^2 \leq mn$$

□

You may reasonably expect that it's "tougher" to require that a graph has no triangles than to require it has no 4-cycles, since triangles have a higher edge to vertex ratio. But no:

BONUS 0.11. If $G \not\supset C_4$, then $m = O(n^{3/2})$.

We say $a_n = O(b_n)$ if there is some C such that for all sufficiently large n, $|a_n| \leq C|b_n|$. We also say $b_n = \Omega(a_n)$.

CH 0.12. Find infinitely many graphs $G \not\supset C_4$ such that $m = \Omega(n^{3/2})$.

Multinomials

Theorem 0.13 (Binomial Theorem)**.**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Is there a trinomial theorem? Yes!

Theorem 0.14 (Trinomial Theorem)**.**

$$(x+y+z)^n = \sum_{\substack{k_i \geq 0 \\ \sum k_i = n}} \binom{n}{k_1, k_2, k_3} x^{k_1} y^{k_2} z^{k_3}$$

where

$$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1! k_2! k_3!}$$

Recall that $0! = 1$, which is important for interpreting the above properly. Turns out we can generalize more:

Theorem 0.15 (Multinomial Theorem)**.**

$$(x_1 + \ldots + x_r)^n = \sum_{\substack{k_i \geq 0 \\ \sum k_i = n}} \binom{n}{k_1, \ldots, k_r} \prod_{i=1}^{r} x_i^{k_i}$$

where

$$\binom{n}{k_1, \ldots, k_r} = \frac{n!}{\prod k_i!}$$

DO 0.16. Prove this

HW 0.17. Count the terms in the sum, aka find the number of solutions to $\sum k_i = n$. This answer should be a simple expression in terms of n and r.

Theorem 0.18. *For a tree, $n = m - 1$.*

Remark 0.19. *Maximal* means cannot be extended while *maximum* means largest possible.

Lemma 0.20. *If $n \geq 2$ then the tree has a vertex of degree 1.*

Proof. Take a *maximal* path. Then we claim that the endpoints have degree 1 in the tree.

DO 0.21. Prove this claim

□

Now we prove our theorem:

Proof. We induct on n. The $n = 1$ case is clear. Assume $n \geq 2$ and pick a vertex x of degree 1. Let $T' = T - x$. Applying the inductive hypothesis to T', we get

$$m_T = m_{T'} + 1 = n_{T'} - 1 + 1 = n_T - 1$$

□

Wait a second let's double check that T' is a tree. It's cycle free since removing edges creates no cycles, but is it connected? Hmm yeah it is, x can't be in any path containing two points in T', since it'd have to be an interior point, and thus have degree at least 2.

Question 0.22. *Given a set V of n vertices, how many graphs are there on V?*

We have to take each pair of vertices and decide whether they are adjacent. So this gives $2^{\binom{n}{2}}$ possibilities. Hmm, how many are trees?

Theorem 0.23 (Cayley's Formula)**.** *There are n^{n-2} trees on n vertices!*

Theorem 0.24. *The number of trees with prescribed degreees is $\frac{(n-2)!}{(\prod d_i - 1)!}$*

DO 0.25. Prove this theorem by induction.

HW 0.26. Use the theorem to prove Cayley's formula.

DO 0.27. Study Prüfer's code on Wikipedia, which gives a bijective proof.

DO 0.28. G is a tree if and only if for any pair of vertices, there's a unique path from one to the other.

DO 0.29. If a connected graph, every pair of largest paths share a vertex.

BONUS 0.30. In a tree, all paths share a vertex.

CH 0.31 (Unlimited time)**.** Show that this isn't true for all connected graphs.

Definition 0.32. If $H = (W, F)$ is a subgraph of $G = (V, E)$, then it is a *spanning subgraph* if $W = V$.

Theorem 0.33. *G has a spanning tree $\iff$ it's connected.*

We prove this theorem by giving a (greedy) algorithm constructing it.
Let $E = \{e_1, \ldots, e_m\}$ and write $e_i = \{u_i, v_i\}$. The desired spanning tree will be (V, F).
<u>initialize</u> $F = \emptyset$
<u>for</u> $i = 1$ <u>to</u> m
  <u>if</u> u_i and v_i are not in the same component of (V, F)
  <u>then</u> $F \Longleftarrow F \cup \{e_i\}$
<u>end</u> (<u>for</u>)
<u>return</u> (V, F)

DO 0.34. Prove this algorithm produces a spanning tree if G is connected.

In this case the greedy algorithm works quite nicely. This isn't always the case.

Graph Coloring

Definition 0.35. A legal coloring of a graph is a map $f : V \to \{\text{colors}\}$ such that if $u \sim v$, then $f(u) \neq f(v)$.

The greedy coloring algorithm is simple: for $v \in V$ use the first available color.

Definition 0.36. The *chromatic number* of a graph, denoted $\chi(G)$, is the minimum number of colors needed for a legal coloring.

HW 0.37. Find G with an even number of vertices such that $\chi(G) = 2$ but the greedy algorithm uses $\frac{n}{2}$ colors.

DO 0.38. For any graph, $\chi(G) \leq \Delta + 1$ where Δ is the maximum degree of a vertex.

DO 0.39. Study asymptotic notation from the lecture notes on the website.