Graph Theory: CMSC 27530/37530 Lecture 1

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Revised by instructor

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Administrative: Send emails to both laci AT cs.(our school) and lbabai AT (google-email). Homeworks should be typeset in LaTeX(and printed), with 2-week grace period to learn LaTeX.

Definition 1.1. A **graph** is a pair of sets, $G = (V, E)$, where V is the set of vertices (nodes) and E is a set of edges (links). An edge is an unordered pair of vertices.

We say $u, v \in E$ are **adjacent** if $\{u, v\} \in E$. The adjacency relation is

1. irreflexive: $\forall u \in V$, we have $u \not\sim u$

2. symmetric: $\forall u, v \in V$, $u \sim v \iff v \sim u$.

Definition 1.2. If $G = (V, E)$ is a graph, the **complement** of G is $\overline{G} = (V, \overline{E})$ where $\overline{E} = \binom{V}{2} \setminus E$. The notation $\binom{V}{2}$ denotes the set of unordered pairs of vertices.

An alternative formulation of the complement is $(\forall u, v \in V)($ if $u \neq v$ then $u \sim_{\overline{G}} v \iff u \not\sim_G v)$.

Notation 1.3. We denote $n = |V|$ to be the **order** of the graph (the number of vertices), and $m = |E|$ to be the **size** of the graph (the number of edges).

Notation 1.4. If n is a positive integer then we write $[n] = \{1, 2, \ldots, n\}$.

Example 1.5. K_n is the complete graph on n vertices, $K_n = \left([n], \binom{[n]}{2}\right)$. We note that for any graph, we have $0 \leq m \leq \binom{n}{2}$, and K_n is the only graph for which $m = \binom{n}{2}$. A complete graph K_n is also called a clique. Observe that $\overline{K_n}$ is the empty graph on n vertices.

Example 1.6. P_n is the path of length $n - 1$. The subscript n denotes the number of vertices, not edges.

Example 1.7. C_n is the cycle on n vertices.

Example 1.8. The $k \times \ell$ grid, denoted $Grid(k, \ell)$. Then $n = k \cdot \ell$. Edges are given by horizontal and vertical adjacencies.

DO 1.1. Determine m for $Grid(k, \ell)$.

Example 1.9. Complete bipartite graphs, $K_{r,s}$. Then $n = r + s$. $m = rs$.

Example 1.10. d-dimensional cube Q_d. $V(Q_d) = \{0, 1\}^d$. $n = 2^d$. Two vertices are adjacent if they differ in exactly one coordinate. $m = d \cdot 2^{d-1}$.

If $u \sim v$, we say u, v are **neighbors**. We denote $N_G(u)$ the set of all neighbors of u. We denote $\deg(u) = |N_G(u)|$ the degree of u (number of neighbors).

Notation 1.11. For a set A, we denote $|A|$ the number of elements of A.

Theorem 1.12 (Handshake Theorem)**.** *For a graph $G = (V, E)$, the following always holds.*

$$\sum_{v \in V} \deg(v) = 2m.$$

DO 1.2. Prove the Handshake Theorem.

DO 1.3. Review relations, equivalence relations.

Definition 1.13. A graph G is **regular** of degree r if $(\forall v)(\deg(v) = r)$.

We note that the d-dimensional cube is regular of degree d, and K_n is regular of degree $n - 1$. Also P_1 and P_2 are regular.

Definition 1.14. Let $G = (V, E)$ and $H = (W, F)$ be graphs. A bijection $f : V \to W$ is an **isomorphism** if f preserves the adjacency relation. Specifically,

$$(\forall u, v \in V)(u \sim_G v \iff f(u) \sim_H f(v)).$$

Definition 1.15. Graphs G, H are **isomorphic** if there exists an isomorphism $f : G \to H$. We write $G \cong H$.

To prove isomorphism, it suffices to find a suitable function f. To prove, non-isomorphism, we typically look for invariants, which are properteis of a graph preserved under isomorphism. The number of edges or vertices are examples of invariants under isomorphism. However, there is no complete set of invariants.

DO 1.4. Show that the two graphs in Figure 1 are isomorphic to one another. They are two representations of the **Petersen graph.**

Definition 1.16. A graph is **self-complementary** if it is isomorphic to its complement.

Some examples of self-complementary graphs are P_1 (a single vertex), P_4, and C_5.

HW 1.1. DUE Thursday **(3 points)** If $G \cong \overline{G}$ then $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$.

The following converse is also true: If $n \equiv 0$ or $1 \mod 4$ then there exists a self-complementary graph with n vertices. — Don't prove this, unless you really want to.

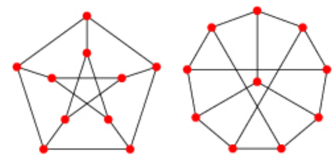

Figure 1: The Petersen Graph in two equivalent representations.

Definition 1.17. A graph $H = (W, F)$ is a **subgraph** of $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$. We write $H \subseteq G$.

Examples of subgraphs include $C_5 \subseteq K_5$ and $K_4 \subseteq K_5$.

HW 1.2. DUE Thursday **(4 points)** Count the 4-cycles (C_4 subgraphs) in Q_d.

Example 1.18. What is the number of triangles (C_3 subgraphs) in K_n? Choose any three vertices. The answer is $\binom{n}{3}$.

Example 1.19. What is the number of 4-cycles in K_n? First, choose any four vertices. Then there are three ways to make a cycle out of four vertices (why?). So the answer is $\binom{n}{4} \cdot 3 = n(n-1)(n-2)(n-3)/8$.

Theorem 1.20 (Mantel–Turán)**.** *If G is triangle-free (meaning $K_3 \not\subseteq G$) then $m \leq \frac{n^2}{4}$.*

DO* 1.5. Prove this theorem. Do not look up solution. Hint: use induction.

DO 1.6. Show that $\forall n$ there exists a triangle-free graph with $m = \lfloor \frac{n^2}{4} \rfloor$.

Definition 1.21. For $u, v \in V$ we say v is **accessible** from u if $\exists u....v$ path.

DO 1.7. Show that accessibility is an equivalence relation. Why is this not obvious? (Show a rigorous proof for transitivity.)

Definition 1.22. The equivalence classes of the accessibility relation are called the connected components. We denote the number of connected components by $k(G)$.

DO 1.8. Show that $m \geq n - k(G)$.

We say that G is connected if $k(G) = 1$. In particular, if G is connected then $m \geq n - 1$.

HW 1.3. DUE Thursday **(5 points)** Given n, what is the maximum m such that $\exists G$ with n vertices and m edges which is disconnected?

Definition 1.23. A **tree** is a connected, cycle-free graph.

Note that unlike the usage in CS, we do not specify a root vertex.

DO 1.9. If T is a tree then $m = n - 1$.

HW 1.4. DUE Thursday **(5 points)** (lose 2 points for each mistake) Draw all 7-vertex trees up to isomorphism. State how many there are. Note that there are two possible mistakes: incompleteness, and duplicate isomorphic copies.