Graph Theory: CMSC 27530/37530 Lecture 2

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April 4, 2019

Definition 2.1. The **Fibonacci Numbers** are the numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

We say that a sequence $a = (a_0, a_1, \dots)$ is of **Fibonacci type** if $a_n = a_{n-1} + a_{n-1}$ for $n \geq 2$.

Definition 2.2. A **geometric progression** is a sequence of the form $(a, aq, aq^2, aq^3, \dots)$.

DO 2.3. For $a \neq 0$, this sequence is Fibonacci type if and only if $q = q_{1,2} = (1 \pm \sqrt{5})/2 \approx 1.618$ and -0.618.

The number $q_1 = (1 + \sqrt{5})/2$ is the *golden ration.*

DO 2.4. Every Fib-type sequence is a linear combination of the sequences $(1, q_1, q_1^2, \dots)$ and $(1, q_2, q_2^2, \dots)$, i. e., $\forall$ Fib-type seq. (a_n) $\exists \alpha, \beta \in \mathbb{R}$ s.t. $a_n = \alpha q_1^n + \beta q_2^n$. (Here q_1 and q_2 are the values from the previous exercise.)

DO 2.5. Find α, β for $a_n = F_n$.

Corollary 2.6. *The Fibonacci Numbers have the following form.*

$$F_n = \left\lfloor \frac{1}{\sqrt{5}} \cdot \left(\frac{1 + \sqrt{5}}{2} \right)^n \right\rceil$$

where the half-brackets indicate "nearest integer."

DO 2.7. Prove previous corollary.

Theorem 2.8 (Mantel–Turán)**.** *If G is a triangle-free graph, then $m \leq n^2/4$.*

Definition 2.9. If a_n, b_n are sequences, we say a_n, b_n are **asymptotically equal** if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$

In this case we write $a_n \sim b_n$.

Lemma 2.10. *If G is triangle free and $x, y \in V(G)$ are adjacent, then* $\deg(x) + \deg(y) \le n$.

Proof. Every $u \in V$ is adjacent to at most one of x, y. □

First proof of Mantel–Turán. By induction on n. Inductive step: If $m = 0$ we are done. Otherwise pick an edge xy. Remove the edge and let G' be the resulting graph. We have

$$m_G \le 1 + (n-2) + m_{G'}.$$

(Why $(n-2)$? Because the edge xy is being counted twice.) By the inductive hypothesis $m_{G'} \le \frac{(n-2)^2}{4}$. Therefore we have

$$m_G \le 1 + (n-2) + \frac{(n-2)^2}{4} = \frac{n^2}{4}.$$

Note that this argument requires two base cases: $n = 1$ and $n = 2$, These cases are obvious. □

Definition 2.11. If $x_1, \dots, x_n \in \mathbb{R}$ then the **arithmetic mean** is defined as

$$A(x_1, \dots, x_n) = \frac{x_1 + \dots + x_n}{n}.$$

Definition 2.12. If $x_1, \dots, x_n \in \mathbb{R}$ then the **quadratic mean** is defined as

$$Q(x_1, \dots, x_n) = \sqrt{\frac{x_1^2 + \dots + x_n^2}{n}}.$$

DO 2.13. Show $A \le Q$; equality holds if and only if $x_1 = \ldots = x_n$.

Second proof of Mantel–Turán. Let's add up the m inequalities stated in the Lemma (one inequality per edge).

$$mn \ge \sum_{\{x,y\} \in E} (\deg(x) + \deg(y)) = \sum_{x \in V} (\deg(x))^2$$

On the other hand, using the inequality between the Arithmetic and the Quadratic mean, we obtain the relation

$$\frac{(2m)^2}{n} = \frac{(\sum \deg(x))^2}{n} \le \sum_{x \in V} (\deg(x))^2 \le mn.$$

Conclude that $m \le \frac{n^2}{4}$. □

BONUS 2.14. (5 points) Prove: If $C_4 \not\subseteq G$ then $m = O(n^{3/2})$. Specifically, $m \le \frac{n^{3/2}}{2} + \frac{n}{4}$.

Notation 2.15 (Big-Oh notation)**.** For two sequences a_n, b_n we write $a_n = O(b_n)$ and say that a_n **is** big-Oh of b_n if

$$(\exists C)(\forall \text{ sufficiently large } n)(\, |a_n| \le C \cdot |b_n|\,).$$

C is called the *implied constant.*

CH 2.16. (12 points) Show that the upper bound in Bonus problem 2.14 is tight, up to a constant factor. Specifically, there exist infinitely many graphs G not containing C_4 s.t. $m_G = \Omega(n_G^{3/2})$.

Notation 2.17 (Big-Omega notation)**.** The Ω notation is the inverse of the big-Oh notation: $b_n = \Omega(a_n)$ if $a_n = O(b_n)$.

Remark 2.18. Challenge problems have no deadline.

Theorem 2.19 (Binomial Theorem)**.**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Theorem 2.20 (Trinomial Theorem)**.**

$$(x+y+z)^n = \sum_{k_1,k_2,k_3 \geq 0 : \sum k_i = n} \binom{n}{k_1, k_2, k_3} x^{k_1} y^{k_2} z^{k_3}$$

where the trinomial coefficient $\binom{n}{k_1,k_2,k_3}$ *is defined by*

$$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1! \cdot k_2! \cdot k_3!}$$

Theorem 2.21 (Multinomial Theorem)**.**

$$(x_1 + \cdots + x_r)^n = \sum_{k_1,\dots,k_r \geq 0 : \sum k_i = n} \binom{n}{k_1, \dots, k_r} x_1^{k_1} \dots x_r^{k_r}$$

where the multinomial coefficient $\binom{n}{k_1,\dots,k_r}$ *is defined by*

$$\binom{n}{k_1, \dots, k_r} = \frac{n!}{\prod k_i!}$$

DO 2.22. Prove the Multinomial Theorem.

HW 2.23. (3 points) Count the terms in the Multinomial Theorem, i. e., how many solutions does the equation $\sum_{i=1}^{r} k_i = n$ have in integers $k_1, \dots, k_r \geq 0$? Hint: answer is very simple expression in terms of n, r involving a binomial coefficient.

Recall: a tree is a connected, cycle-free graph.

Theorem 2.24. *For a tree,* $m = n - 1$*.*

Lemma 2.25. *Every tree with* $n \geq 2$ *vertices has a vertex of degree* 1*.*

Terminology 2.26 (Maximal vs. maximum)**.** An object is **maximal** if it cannot be extended. An object is **maximum** if it is the largest among all objects under consideration.

Proof. Take any maximal path. Claim: Its endpoints have degree 1. □

DO 2.27. Prove the claim.

Proof of theorem. We proceed by induction on n. The case $n = 1$ is a base case. Inductive step: let T be a tree with $n \geq 2$. Pick a vertex x of degree 1. Let T' be the tree obtained by removing x. Apply IH to T', so we get $m' = n' - 1$. But $m' = m - 1$ and $n' = n - 1$, so $(m - 1) = (n - 1) - 1$, i. e., $m = n - 1$, as desired.

To justify our use of the IH, we need to show: T' is still a tree.

Claim 1: T' is cycle free (trivial). Claim 2: T' is connected.

To show Claim 2, let $u, v \in V_{T'}$. We need to show $\exists u \ldots v$ path in T'. We know such a path P exists in T. We claim that $x \notin V(P)$. Indeed, x cannot be an endpoint of P, since $x \neq u, v$. Moreover, x cannot be an interior point of the path, because x has degree 1. □

Remark 2.28. Do not forget to show that the assumptions of the IH hold for the smaller object to which you are trying to apply the IH. (In the above case, we had to show that T' was still a tree).

Given a set V of n vertices what is the number of graphs on V? The anwer is $2^{\binom{n}{2}}$.

How many among them are trees? The remarkably simple answer is given by Cayley's Formula.

Theorem 2.29 (Cayley's Formula)**.** *The number of trees on a given set of n vertices is*

$$n^{n-2}$$

How can we prove Cayley's Formula? We outline one approach.

Theorem 2.30 (Counting trees with prescribed degrees)**.** *Let $d_1, \ldots, d_n \geq 1$ be integers such that $\sum d_i = 2n - 2$. The number of trees with vertex set $[n]$ such that $\deg(i) = d_i$ is*

$$\frac{(n-2)!}{\prod (d_i - 1)!}.$$

DO 2.31. Prove this theorem. Hint: induction on n.

HW 2.32. (3 points) Use Theorem 2.30 to prove Cayley's Formula.

An alternative proof of Cayley's Formula is a *bijective proof* called **Prüfer's Code.**

DO 2.33. Study Prüfer's Code.

DO 2.34. G is a tree if and only if $(\forall u, v \in V)(\exists! u \ldots v \text{ path})$ (the exclamation point means "unique").

DO 2.35. In a connected graph, every pair of longest (maximum length) paths share a vertex.

BONUS 2.36. (12 points) In a tree, all longest paths share a vertex.

CH 2.37. (15 points) Show that the previous statement is not true for all connected graphs.

Definition 2.38. A subgraph $H \subseteq G$ is called a **spanning subgraph** if $V = W$ (no vertices removed). H is moreover called a **spanning tree** if H is a tree.

Theorem 2.39. *A graph G has a spanning tree if and only if G is connected.*

Proof. We prove the theorem by a **greedy algorithm**. Let $E = \{e_1, \dots, e_m\}$, and $e_i = \{u_i, v_i\}$. The desired spanning tree will be (V, F).

initialize $F := \varnothing$.
for $i = 1$ **to** n:
  if u_i and v_i are not in the same component of (V, F) **then** $F \leftarrow F \cup \{e_i\}$
end(for)
return (V, F) □

DO 2.40. Prove that this algorithm produces a spanning tree assuming G is connected. What does it do if G is not connected?

Definition 2.41. A **legal coloring** of a graph is a mapping $f : V \to \{\text{colors}\}$ such that if $u \sim v$ then $f(u) \neq f(v)$. The goal is to use as few colors as possible.

A **greedy coloring algorithm:**

for $v \in V$ use the first available color (the colors are ordered).

Definition 2.42. The **chromatic number** of a graph is the minimum number of colors needed for a legal coloring. We denote this number $\chi(G)$.

HW 2.43 (Dismal failure of greedy coloring)**. (4 points)** For every even value of the positive integer n, find a graph G with vertex set $V = [n]$ such that $\chi(G) = 2$ but the greedy coloring requires $n/2$ colors.

DO 2.44 (Success of greedy coloring)**.** Prove: $(\forall G)(\chi(G) \leq \Delta + 1)$. Here Δ denotes the maximum degree in G.
Use greedy coloring for the proof.

DO 2.45. For **asymptotic notation** (asymptotic equality, big-Oh, big-Omega) study the instructor's **"Discrete Mathematics" online lecture notes** (linked among Texts/Online references on the course home page).