Recall: a legal coloring for a graph is a mapping \( f : V \rightarrow \{\text{colors}\} \) such that \((\forall x, y \in V)(x \sim y \Rightarrow f(x) \neq f(y))\). The minimum number of colors necessary for a legal coloring is called the **chromatic number**, denoted \(\chi(G)\). If \(\chi(G) \leq k\), we say that \(G\) is \(k\)-colorable.

**Definition 3.1.** A graph \(G\) is bipartite if it is 2-colorable, i.e., if \(\chi(G) \leq 2\).

In other words, \(G = (V, E)\) is bipartite if \(V\) can be split as \(V = V_1 \sqcup V_2\) (disjoint union) such that all edges connect \(V_1\) to \(V_2\). Thinking of the edges as ordered rather than unordered pairs, in this case we can write \(E \subseteq V_1 \times V_2\).

**DO 3.2.** Show that \(\chi(G) = n\) if and only if \(G\) is complete.

**Example 3.3.** The graph Grid\((k, \ell)\) is bipartite. Color vertices in the style of a chess board.

Which cycles \(C_n\) are bipartite?

**DO 3.4.** Show that \(C_n\) is bipartite if and only if \(n\) is even.

If \(H \subseteq G\), then \(\chi(H) \leq \chi(G)\) because every legal coloring of \(G\) gives a legal coloring when restricted to \(H\).

**DO 3.5.** \(G\) is bipartite if and only if \(G\) has no odd cycles.

How can we characterise 3-colorability? Clearly \(K_4 \not\subseteq G\). Is there a graph \(G\) that is not 3-colorable but does not contain \(K_4\)? Try to construct such a graph.

First observation: If an example works but it has a vertex of degree 2, then that vertex can be deleted.

**Definition 3.6.** The **clique number** of a graph \(G\) is \(\max\{t : K_t \subseteq G\}\). We denote the clique number by \(\omega(G)\).

Can we find graphs with \(\chi(G) > \omega(G)\)? The pentagon \((C_5)\) is an example. Now can we raise the chromatic number?

**Definition 3.7.** The **cone** of the graph \(G\) is obtained from \(G\) by adding a vertex and joining that vertex to all vertices of \(G\).
DO 3.8. Let \( G' \) be the cone of \( G \). Then \( \omega(G') = 1 + \omega(G) \) and \( \chi(G') = 1 + \chi(G) \).

The cone of \( C_5 \) is the wheel \( W_6 \) with 5 spokes. So \( \chi(W_6) = 4 \) and \( \omega(W_6) = 3 \).

Theorem 3.9. \((\forall k)(\exists G)(G \not\supset K_3 \text{ and } \chi(G) \geq k)\).

HW 3.10. DUE Tuesday (4 points) Draw a triangle-free graph with \( \chi(G) \geq 4 \) using as few vertices as possible. You don’t need to prove that the graph is triangle-free, but do sketch the proof why your graph is not 3-colorable. Hint: \( n = 11 \), and your drawing of the graph should have 5-fold rotational symmetry. (No collaboration whatsoever is allowed on this problem. Don’t look it up on the web. Have fun.)

Notation: \( \Delta = \Delta(G) \) is the maximum degree of \( G \) (maximum of the degrees of the vertices). Lecture 2 gave the result that \( \chi(G) \leq \Delta + 1 \) (the proof is by greedy algorithm).

Theorem 3.11 (Brooks’s Theorem). If \( G \) is connected and \( \chi(G) = \Delta + 1 \), then \( G \) is either a clique or an odd cycle.

HW 3.12. DUE Tuesday (5 points) If \( G \) is triangle-free, then \( \chi(G) = O(\sqrt{n}) \). Give a specific upper bound (the bound should not involve big-Oh notation), and state your implied constant.

Definition 3.13. For a set of colors \( [x] = \{1, ..., x\} \), the number of legal colorings on \( G \) is denoted \( f_G(x) \).

For a positive integer \( x \), \( f_G(x) > 0 \iff x \geq \chi(G) \).

Example 3.14. For the empty graph, \( f_{K_n}(x) = x^n \).

Example 3.15. For the complete graph, \( f_{K_n}(x) = x(x - 1) \ldots (x - n + 1) \).

BONUS 3.16. DUE Tuesday (5 points) Prove that for any \( G \), \( f_G(x) \) is a polynomial of \( x \). The function is called the chromatic polynomial.

A graph is planar if it can be drawn in the plane without intersecting edges.

Theorem 3.17 (4-Color Theorem (4CT)). If \( G \) is planar, then it is 4-colorable.

Theorem 3.18. The graphs \( K_5 \) and \( K_{3,3} \) are not planar.

A restatement of the 4CT: If \( G \) is planar then \( f_G(4) > 0 \).
Theorem 3.19 (Tutte). If $G$ is planar, then $f_G(2 + \gamma) > 0$, where $\gamma \approx 1.618$ is the golden ratio.

Definition 3.20. For a given graph $G$, a subset $A \subseteq G$ is an independent set if there is no edge within $A$. In other words, $E \cap \binom{A}{2} = \emptyset$.

Definition 3.21. The independence number $\alpha(G)$ is the maximum size of an independent set in $G$.

DO 3.22. Show that $\alpha(P_n) = \lceil \frac{n}{2} \rceil$.

DO 3.23. Show that $\alpha(C_n) = \lceil \frac{n}{2} \rceil$.

HW 3.24. DUE Tuesday (4 points) Prove that $\alpha(\text{Grid}(k, \ell)) = \lceil \frac{k\ell}{2} \rceil$. (Showing that a $\lceil \frac{k\ell}{2} \rceil$ independent set exists is the easy part. Prove that there is no larger independent set. The proof should be simple and convincing. Elegance counts.)

DO 3.25. Show that $\alpha(G) = \omega(G)$.

HW 3.26. DUE Tuesday (4 points) Show that $\alpha(G) \cdot \chi(G) \geq n$.

Definition 3.27. The cartesian product graphs $G = (V, E)$ and $H = (W, F)$ is a graph, denoted $L = G \square H$, with vertices $V(L) = V(G) \times V(H)$. A vertex $(x_1, y_1)$ is adjacent to a vertex $(x_2, y_2)$ either of the following conditions hold:

1. $x_1 = x_2$ and $y_1 \sim_H y_2$
2. $y_1 = y_2$ and $x_1 \sim_G x_2$.

Notice that $\text{Grid}(k, \ell) = P_k \square P_\ell$.

The torus is the surface of the donut shape. We call the graph $C_k \square C_\ell$ the $k \times \ell$ toroidal grid (the infinite square grid wrapped around a torus). (See fig. 2.) It is a 4-regular graph with $k\ell$ vertices.

![Figure 2: The toroidal grid. Image: mathoverflow “Torus Graph Dynamics”](image-url)
HW 3.28. DUE Tuesday (4+4 points)

(a) Determine $\alpha(C_{13} \square C_{13})$.
(b) Determine $\alpha(C_{13} \square C_{29})$.

Elegance counts when proving the upper bounds.

DO 3.29. What is the size of the smallest maximal independent set in $P_n$? (To be maximal means that no vertex can be added and still be an independent set.) Show that the answer is $[\frac{n}{2}]$.

DO 3.30. Do the same for $C_n$.

HW 3.31. DUE Thursday (3 points) Given $n$, find

$$\max \left\{ \frac{\alpha(G)}{|A|} : G \text{ has } n \text{ vertices and } A \text{ is a maximal independent set} \right\}.$$

HW 3.32. DUE Thursday (4 points) If $G$ is regular and non-empty then $\alpha(G) \leq n/2$. Look for a short, convincing argument.

DO 3.33. If $K_{2,3} \not\subset G$, then $m = O(n^{3/2})$.

DO 3.34. If $K_{3,3} \not\subset G$, then $m = O(n^{5/3})$.

Definition 3.35. A weighted graph is a triple $G = (V, E, w)$, where $w : E \to \mathbb{R}$ maps edges to “weights.”

DO 3.36. Show that the greedy algorithm succeeds at producing a minimum-weight spanning tree, when the edges are sorted in non-decreasing order of weight. (The weight of a tree is the sum of the edge weights).