Graph Theory: CMSC 27530/37530 Lecture 3

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Revised by instructor

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Recall: a legal coloring for a graph is a mapping $f : V \to \{\text{colors}\}$ such that $(\forall x, y \in V)(x \sim y \Rightarrow f(x) \neq f(y))$. The minimum number of colors necessary for a legal coloring is called the **chromatic number**, denoted $\chi(G)$. If $\chi(G) \leq k$, we say that G is k**-colorable**.

Definition 3.1. A graph G is bipartite if it is 2-colorable, i. e., if $\chi(G) \leq 2$.

In other words, $G = (V, E)$ is bipartite if V can be split as $V = V_1 \sqcup V_2$ (disjoint union) such that all edges connect V_1 to V_2. Thinking of the edges as ordered rather than unordered pairs, in this case we can write $E \subseteq V_1 \times V_2$.

DO 3.2. Show that $\chi(G) = n$ if and only if G is complete.

Example 3.3. The graph $\text{Grid}(k, \ell)$ is bipartite. Color vertices in the style of a chess board.

Which cycles C_n are bipartite?

DO 3.4. Show that C_n is bipartite if and only if n is even.

If $H \subseteq G$, then $\chi(H) \leq \chi(G)$ because every legal coloring of G gives a legal coloring when restricted to H.

DO 3.5. G is bipartite if and only if G has no odd cycles.

How can we characterise 3-colorability? Clearly $K_4 \not\subseteq G$. Is there a graph G that is not 3-colorable but does not contain K_4? Try to construct such a graph.

First observation: If an example works but it has a vertex of degree 2, then that vertex can be deleted.

Definition 3.6. The **clique number** of a graph G is $\max\{t : K_t \subseteq G\}$. We denote the clique number by $\omega(G)$.

Can we find graphs with $\chi(G) > \omega(G)$? The pentagon (C_5) is an example. Now can we raise the chromatic number?

Definition 3.7. The **cone** of the graph G is obtained from G by adding a vertex and joining that vertex to all vertices of G.

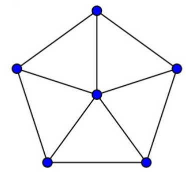

Figure 1: The 5-spoked wheel W_6. Image: Wikipedia

DO 3.8. Let G' be the cone of G. Then $\omega(G') = 1 + \omega(G)$ and $\chi(G') = 1 + \chi(G)$.

The cone of C_5 is the **wheel** W_6 with 5 spokes. So $\chi(W_6) = 4$ and $\omega(W_6) = 3$.

Theorem 3.9. *$(\forall k)(\exists G)(G \not\supset K_3$ and $\chi(G) \geq k)$.*

HW 3.10. DUE Tuesday **(4 points)** Draw a triangle-free graph with $\chi(G) \geq 4$ using as few vertices as possible. You don't need to prove that the graph is triange-free, but do sketch the proof why your graph is not 3-colorable. Hint: $n = 11$, and your drawing of the graph should have 5-fold rotational symmetry. (No collaboration whatsoever is allowed on this problem. Don't look it up on the web. Have fun.)

Notation: $\Delta = \Delta(G)$ is the maximum degree of G (maximum of the degrees of the vertices). Lecture 2 gave the result that $\chi(G) \leq \Delta + 1$ (the proof is by greedy algorithm).

Theorem 3.11 (Brooks's Theorem)**.** *If G is connected and $\chi(G) = \Delta + 1$, then G is either a clique or an odd cycle.*

HW 3.12. DUE Tuesday **(5 points)** If G is triangle-free, then $\chi(G) = O(\sqrt{n})$. Give a specific upper bound (the bound should not involve big-Oh notation), and state your implied constant.

Definition 3.13. For a set of colors $[x] = \{1, ..., x\}$, the number of legal colorings on G is denoted $f_G(x)$.

For a positive integer x, $f_G(x) > 0 \iff x \geq \chi(G)$.

Example 3.14. For the empty graph, $f_{\overline{K}_n}(x) = x^n$.

Example 3.15. For the complete graph, $f_{K_n}(x) = x(x-1)\dots(x-n+1)$.

BONUS 3.16. DUE Tuesday **(5 points)** Prove that for any G, $f_G(x)$ is a polynomial of x. The function is called the **chromatic polynomial**.

A graph is **planar** if it can be drawn in the plane without intersecting edges.

Theorem 3.17 (4-Color Theorem (4CT))**.** *If G is planar, then it is 4-colorable.*

Theorem 3.18. *The graphs K_5 and $K_{3,3}$ are not planar.*

A restatement of the 4CT: If G is planar then $f_G(4) > 0$.

Theorem 3.19 (Tutte)**.** *If G is planar, then $f_G(2+\gamma) > 0$, where $\gamma \approx 1.618$ is the golden ratio.*

Definition 3.20. For a given graph G, a subset $A \subseteq G$ is an **independent set** if there is no edge within A. In other words, $E \cap \binom{A}{2} = \varnothing$.

Definition 3.21. The **independence number** $\alpha(G)$ is the maximum size of an independent set in G.

DO 3.22. Show that $\alpha(P_n) = \lceil \frac{n}{2} \rceil$.

DO 3.23. Show that $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$.

HW 3.24. DUE Tuesday **(4 points)** Prove that $\alpha(\text{Grid}(k,\ell)) = \lceil \frac{k\ell}{2} \rceil$. (Showing that a $\lceil \frac{k\ell}{2} \rceil$ independent set exists is the easy part. Prove that there is no larger independent set. The proof should be simple and convincing. Elegance counts.)

DO 3.25. Show that $\alpha(G) = \omega(\overline{G})$.

HW 3.26. DUE Tuesday **(4 points)** Show that $\alpha(G) \cdot \chi(G) \geq n$.

Definition 3.27. The **cartesian product** graphs $G = (V, E)$ and $H = (W, F)$ is a graph, denoted $L = G \,\square\, H$, with vertices $V(L) = V(G) \times V(H)$. A vertex (x_1, y_1) is adjacent to a vertex (x_2, y_2) either of the following conditions hold:

1. $x_1 = x_2$ and $y_1 \sim_H y_2$
2. $y_1 = y_2$ and $x_1 \sim_G x_2$.

Notice that $\text{Grid}(k,\ell) = P_k \,\square\, P_\ell$.

The *torus* is the surface of the donut shape. We call the graph $C_k \,\square\, C_\ell$ the $k \times \ell$ **toroidal grid** (the infinite square grid wrapped around a torus). (See fig. 2.) It is a 4-regular graph with $k\ell$ vertices.

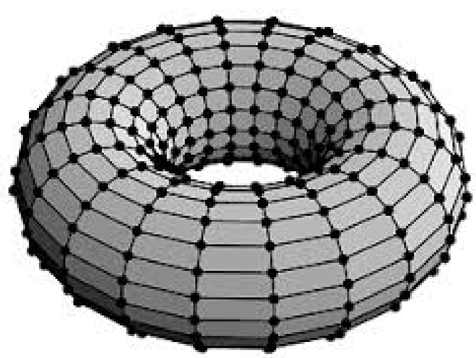

Figure 2: The toroidal grid. Image: math*overflow* "Torus Graph Dynamics"

HW 3.28. DUE Tuesday **(4+4 points)**

(a) Determine $\alpha(C_{13} \,\square\, C_{13})$.

(b) Determine $\alpha(C_{13} \,\square\, C_{29})$.

Elegance counts when proving the upper bounds.

DO 3.29. What is the size of the smallest *maximal* independent set in P_n? (To be *maximal* means that no vertex can be added and still be an independent set.) Show that the answer is $\lceil \frac{n}{3} \rceil$.

DO 3.30. Do the same for C_n.

HW 3.31. DUE Thursday **(3 points)** Given n, find

$$\max\left\{\frac{\alpha(G)}{|A|} : \ G \text{ has } n \text{ vertices and } A \text{ is a maximal independent set}\right\}.$$

HW 3.32. DUE Thursday **(4 points)** If G is regular and non-empty then $\alpha(G) \leq n/2$. Look for a short, convincing argument.

DO 3.33. If $K_{2,3} \not\subseteq G$, then $m = O(n^{3/2})$.

DO 3.34. If $K_{3,3} \not\subseteq G$, them $m = O(n^{5/3})$.

Definition 3.35. A **weighted graph** is a triple $G = (V, E, w)$, where $w : E \to \mathbb{R}$ maps edges to "weights."

DO 3.36. Show that the greedy algorithm succeeds at producing a minimum-weight spanning tree, when the edges are sorted in non-decreasing order of weight. (The weight of a tree is the sum of the edge weights).