Graph Theory: CMSC 27530/37530 Lecture 5

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Recall a previous **Bonus problem 2.36:** In a tree, all longest paths share a vertex.

Several attempted proofs could be streamlined by using the fact that in a tree, between every pair of vertices their is a unique path. The next exercise is an application of this fact.

DO 5.1. The intersection of any number of subtrees of a tree is a subtree.

Proof. Let $T_1, \ldots, T_k$ be the subtrees and (W, F) their intersection. We need to show that (W, F) is connected. Pick two vertices $u, v \in W$. In the tree, $\exists! u...v$ path P. Since in each T_i there is a $u...v$ path, this path can only be P, so all the T_i share P and therefore P is in the intersection of all the T_i. □

A **first proof** of Problem 2.36 can be based on the following exercise.

DO 5.2. Let $T_1, \ldots, T_k$ be subtrees of a tree. Assume each pair (T_i, T_j) shares a vertex. Then all the T_i share a vertex.

Here is the idea of a **second proof**. If a path has even length, then it has a center vertex. If a path has odd length, then it has a center edge.

DO 5.3. If P_1 and P_2 are longest paths in a tree, then P_2 contains the center of P_1.

So the center of one of the longest paths will be shared by all longest paths.

Definition 5.4. A **walk** of length k in a graph G is a sequence of $k+1$ vertices, $v_0, \ldots, v_k$, such that $(\forall i)(v_{i-1} \sim v_i)$.

Remark 5.5. Unlike a path (which is a subgraph), a walk is directed, from start to end.

DO 5.6. If y is accessible from x by a walk, then y is also accessible from x by a path.

Remark 5.7. This exercise is a way to prove a previous exercise, which says that accessibility is a transitive relation.

The following exercise shows the existence of a path.

DO 5.8. A shortest walk from x to y is always a path.

This raises an algorithmic problem: can we find such a path efficiently?

DO 5.9. Given an $x...y$ walk, find an $x...y$ path efficiently. (The algorithm should be linear time.)

Definition 5.10. A **closed walk** is a walk with $v_0 = v_k$.

DO 5.11. If there exists a closed walk of odd length in a graph then the graph contains an odd cycle. – There are two branches to this problem: (1) existence, and (2) an efficient algorithm for finding the odd cycle.

Definition 5.12. An **Eulerian Trail** is a walk that traverses every edge exactly once. An Eulerian Trail is **closed** if it is also a closed walk. A graph is **Eulerian** if it has a closed Eulerian trail.

When is a graph G Eulerian? Certainly G must be connected. Since every vertex is entered the same number of times it is exited, each vertex must have even degree.

DO 5.13. Show that G is Eulerian if and only if G is connected and all degrees are even.

Definition 5.14. The **adjacency matrix** of a graph with vertices $V = \{v_1, ..., v_n\}$ is the $n \times n$ matrix A with entries (a_{ij}) where

$$a_{ij} = \begin{cases} 1 & i \sim j \\ 0 & i \not\sim j \end{cases}.$$

In particular, $a_{ii} = 0$ and $a_{ij} = a_{ji}$ (A is symmetric). The symmetry of A will have important consequences due to the Spectral Theorem for symmetric matrices.

What is the significance of the matrix A^k, the k-th power of A? Let $a_{ij}^{(k)}$ denote the (i, j)-entry of A^k.

DO 5.15. Show that $a_{ij}^{(k)}$ is the number of $i...j$ walks of length k.

Recall a previous exercise: G is bipartite $\iff$ G has no odd cycle.

One direction is immediate (an odd cycle is not bipartite). To show the other direction, without loss of generality, assume G is connected.

DO 5.16. Why is the previous statement without loss of generality?

DO 5.17. Every tree is bipartite.

Proof. Induction on n. Delete a vertex x of degree 1, color the rest, add x back. Since x has only one neighbor, x can be colored by the color not taken by the neighbor. □

Now to two-color a connected graph G, pick a spanning tree T of G, and two-color T. Does this give a legal coloring of G? If not, we show that the graph has an odd cycle. Indeed, in this case there exist vertices $u \sim v$ having the same color in the tree. (So the edge $\{u, v\}$ does not belong to the tree.) Let P be the unique path in the tree connecting u to v. Along this path, the colors alternate; therefore P has even length (because u and v have the same color). But now by adding the edge $\{u, v\}$ to P we get an odd cycle.

Definition 5.18. A **matching** in a graph G is a set of disjoint edges. (No two of those edges share a vertex.)

Definition 5.19. The maximum size of a matching in a graph G is called the **matching number**, denoted $\nu(G)$. (This is the Greek letter *nu*; in LaTeX, `\nu`.)

For example, $\nu(P_n) = \lfloor \frac{n}{2} \rfloor$, and also $\nu(K_n) = \lfloor \frac{n}{2} \rfloor$. Note that in any graph G, $\nu(G) \le \lfloor \frac{n}{2} \rfloor$.

DO 5.20. Find $\nu(C_n)$.

Definition 5.21. A **cover** in a graph G is a set of vertices which hits every edge.

Definition 5.22. The **covering number** of a graph G is the minimum size of a cover, denoted $\tau(G)$. (This is the Greek letter *tau*; in LaTeX, `\tau`.)

Remark 5.23. The covering number τ has also been called the "transversal number," explaining the τ notation. In computer science, a cover is often called a "hitting set" and τ the "hitting number."

DO 5.24. The complement of a cover is an independent set and vice versa.

DO 5.25. Show that $\alpha + \tau = n$.

DO 5.26. Show that $\nu \le \tau$.

HW 5.27. (Due Thursday) **(3 points)** Show that $\tau \le 2\nu$. Show that, in fact, for every *maximal* matching M,

$$\tau \le 2|M|.$$

DO 5.28. For every k, find a graph G such that $\nu(G) = k$ and $\tau(G) = 2k$.

DO 5.29. Solve the previous problem with the additional requirement that G is connected.

DO 5.30. Find $\tau(P_n)$, $\tau(C_n)$, $\tau(K_n)$.

Since the $\nu \le \tau$ inequality always holds, we are interested in large classes of graphs where we have equality.

Theorem 5.31 (Dénes Kőnig, circa 1913)**.** *For a bipartite graph, $\nu = \tau$.*

Remark 5.32. Dénes Kőnig (1884–1944), a mathematician in Budapest, was the author of the first monograph about graph theory (1936). The letter "ő" in his name is the Hungarian "long Umlaut," for which Donald Knuth provided the TeX macro `\H{o}` so the TeX code for Kőnig's name is `K\H{o}nig`. It is, however, often typeset with the more familiar German Umlaut as "König" (`K\"onig`).

Definition 5.33. Given a matching M, we say that a set $A \subseteq V$ is *matched* by M if $(\forall x \in A)(\exists e \in M)(x \in e)$.

Notation 5.34. The set of neighbors of a vertex x in the graph G is denoted $N_G(x)$, or simply $N(x)$ if G is clear from the context. For a set $A \subseteq V$ we write

$$N(A) = \bigcup_{x \in A} N(x) \setminus A.$$

We call this set the *set of neighbors of* A.

DO 5.35. If A is an independent set then

$$N(A) = \bigcup_{x \in A} N(x).$$

Theorem 5.36 (Philip Hall's "Marriage Theorem")**.** *We are given a bipartite graph with bipartition* $V = L \sqcup R$. *There exists a matching which matches the set* L *if and only if* $(\forall A \subseteq L)(|A| \leq |N(A)|)$.

Definition 5.37. We call a subset $A \subseteq L$ such that $|A| > |N(A)|$ a *Hall obstacle* (to matching W). So Hall's Theorem says that in a bipartite graph, either L can be matched or there exists a Hall obstacle.

HW 5.38. (Due Thursday) **(4 points)** Derive Hall's Theorem from Kőnig's.

Definition 5.39. A **predicate** over a domain X is a function $f : X \to \{0, 1\}$.

Here 1/0 is usually interpreted as yes/no, or true/false.

Definition 5.40. A **decision problem** on a domain X is the problem to compute a predicate on X.

A predicate f is **well-characterized** if both answers have short proofs. Examples of well-characterized predicates:

- Given a graph G, is it bipartite?
  The "yes" answer is demonstrated by a 2-coloring.
  The "no" answer is demonstrated by an odd cycle.
  The good characterization is the theorem that says that there is no third case: a graph is either bipartite or it has an odd cycle.

- Given a graph, is it Eulerian?
  The "yes" answer is demonstrated by a closed Eulerian trail.
  The "no" answer is demonstrated by a proper connected component or a vertex of odd degree.
  The good characterization is Euler's characterization of Eulerian graphs (problem 5.13).

- Given a bipartite graph with bipartition $V = L \sqcup R$, can L be matched?
  The "yes" answer is demonstrated by a matching that matches L.
  The "no" answer is demonstrated by a Hall obstacle.
  The good characterization is Hall's theorem.

- Given a bipartite graph and an integer k, the statement "$\nu \geq k$" is well characterized.
  The "yes" answer is demonstrated by a matching of size k.
  The "no" answer is demonstrated by a cover of size $k-1$.
  The good characterization is Kőnig's Theorem.
  In this case we also say that in bipartite graphs, the quantities ν and τ are well characterized.

A formalization of the concept of "good characterization" is provided by the complexity class $\mathsf{NP} \cap \mathsf{coNP}$.