Graph Theory: CMSC 27530/37530 Lecture 6

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Revised by instructor

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HW 6.1. (5 points) Count the independent sets in P_n. (Note that the empty set is independent. More generally, every subset of an independent set is independent.) The answer has a simple form in terms of a known quantity.

HW 6.2. (4 points) What is the maximum possible number of maximal paths in a tree with n vertices? Your answer should be a simple expression.

HW 6.3. (4 points) For all sufficiently large n, find a connected graph with at least 100^n longest paths.

BONUS 6.4. (5 points) For infinitely many values of n, find a connected, 3-regular graph with n vertices and exponentially many longest paths. "Exponentially many" means more than $(1+c)^n$ for some constant $c > 0$. State your constant. (As usual, n is the number of vertices.)

Recall a previous challenge problem: Find a connected graph where no vertex is shared by all longest paths. — Solutions that have been found so far share the property that any eight longest paths share a vertex, but you can find nine longest paths that do not share a vertex.

CH+ 6.5. Does there exist a connected graph with three longest paths that do not share a vertex?

Definition 6.6. A vertex $v \in V(G)$ is a **cut vertex** if the number of connected components in G increases when v is removed.

Definition 6.7. For $k \geq 1$ we say that a connected graph is **k-connected** if it remains connected when any $k-1$ or fewer vertices are removed. Note that if G is k-connected and $\ell \leq k$ then, by definition, G is also ℓ-connected. — This definition does not apply to the case when the graph is complete since no matter how many vertices we remove from a complete graph, it remains connected. K_n is said to be $n-1$-connected, but not n-connected. (So it is also k-connected for all $k \leq n-1$.) The reason of this convention will be explained later.

CH+ 6.8. Does there exist a 3-connected 3-regular graph where the longest paths do not share a vertex?

Recall the definition of a *finite probability space:* a pair (Ω, P), where Ω is a non-empty finite set called a sample space, and $P : \Omega \to \mathbb{R}$ is a function satisfying

(i) $(\forall x \in \Omega)(P(x) \geq 0)$

(ii) $\sum_{x \in \Omega} P(x) = 1$.

Such a function is called a *probability distribution* over Ω, We say that P is the *uniform distribution* if $(\forall x \in \Omega)(P(x) = \frac{1}{|\Omega|})$.

An event is a subset $A \subseteq \Omega$. For an event A, we define

$$P(A) = \sum_{x \in A} P(x).$$

It follows that $P(\varnothing) = 0$, and $P(\Omega) = 1$. Furthermore, $0 \leq P(A) \leq 1$ for any event A, and $P(\overline{A}) = 1 - P(A)$, where $\overline{A} = \Omega \setminus A$.

DO 6.9 (Union bound)**.** Show that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

A *random variable* on the probability space (Ω, P) is a function $X : \Omega \to \mathbb{R}$.

For a random variable X, we define the *expected value* as $E(X) = \sum_{x \in \Omega} P(x) \cdot X(x)$. In the case P is the uniform distribution,

$$E(X) = \frac{1}{|\Omega|} \sum_{x \in \Omega} X(x)$$

is the arithmetic mean of the values taken by the random variable at each element of the sample space Ω.

Recall a previous DO exercise: $\min X \leq E(X) \leq \max X$.

Theorem 6.10.

$$E(X) = \sum_{y \in \mathbb{R}} y \cdot P(X = y).$$

Recall that a random variable Y is an *indicator variable* if $Y : \Omega \to \{0, 1\}$. There is a 1-1 correspondence between indicator variables and events. For an event A, there is an associated indicator variable ϑ_A defined by

$$\vartheta_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

For an arbitrary indicator variable Y, the event A defined by $A = Y^{-1}(1)$ gives $\vartheta_A = Y$.

DO 6.11. $E(\vartheta_A) = P(A)$.

The most important fact about expectation is that it is *linear*. This means the following. Given random variables $X_1, ..., X_k$ on (Ω, P) and real numbers $c_1, \ldots, c_n$, the expected value of the *linear combination* $\sum_{i=1}^{k} c_i \cdot X_i$ distributes over its terms as follows:

$$E\left(\sum_{i=1}^{k} c_i \cdot X_i\right) = \sum_{i=1}^{k} c_i \cdot E(X_i).$$

Example 6.12. Let us flip n biased coins, with the probability

$$P(i\text{-th coin is heads}) = p_i.$$

Let X be the total number of heads. What is $E(X)$? We can write

$$X = \sum_{i=1}^{n} Y_i$$

where Y_i is the indicator variable indicating that the event that the i-th coin comes up heads. By the linearity of expectation,

$$E(X) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} p_i.$$

Recall that a *permutation* of a set V is a bijection $\pi : V \to V$. If $|V| = n$, then there are $n!$ permutations of the set V.

Lecture 4 stated the Wei-Caro lower bound on the independence number. We now present a proof of this bound.

Proof of Wei-Caro. Recall the *greedy independent set algorithm* for a graph G with vertex set $V = \{v_1, \ldots, v_n\}$:

initialize $I := \varnothing$.
for $i = 1$ **to** n
  if v_i has no neighbor in I **then** $I \leftarrow I \cup \{v_i\}$
end(for)
return I.

Clearly, the set I returned is an independent set and therefore $\alpha(G) \geq |I|$.

We shall use a randomized version of this algorithm: first we randomly permute the vertices and then apply the greedy independent set algorithm. Let X denote the expected size of the independent set we get. We claim that

$$E(X) \geq \sum_{i=1}^{n} \frac{1}{1 + d_i}. \tag{1}$$

Since $\alpha(G) \geq X$ (always), it follows that $\alpha(G) \geq \max X \geq E(X)$, proving the Wei–Caro bound.

Let us formalize and prove these statements. Let Ω be the set of all permutations of $V(G)$, and P be the uniform distribution over Ω. For $\pi \in \Omega$, let $I(\pi)$ denote the independent set obtained by the greedy algorithm after applying the permutationn π to the set of vertices. Let $X(\pi) = |I(\pi)$. So X is a random variable over the probability space (Ω, P). We can write

$$X = \sum_{i=1}^{n} Y_i$$

where Y_i is the indicator variable indicating the event that $i \in I(\pi)$.

We need to estimate

$$P(v_i \in I(\pi)).$$

If v_i is the first among all of its neighbors under permutation π then $v_i \in I(\pi)$. So

$$P(v_i \in I) \geq P\big(\pi(v_i) < \pi(u) \text{ for every } u \in N(v_i)\big) = \frac{1}{1 + d_i}$$

The reason for the rightmost equation is that when all vertices are randomly permuted, then in particular the set $\{v_i\} \cup N(v_i)$ of $1 + d_i$ vertices comes in random order, so each element of this set has an equal chance to come first.

As a result,

$$\begin{aligned} E(X) &= \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} P(v_i \in I(\pi)) \\ &\geq \sum_{i=1}^{n} P\big(\pi(v_i) < \pi(u) \text{ for every } u \in N(v_i)\big) \\ &= \sum_{i=1}^{n} \frac{1}{1 + d_i}. \end{aligned}$$

□

HW 6.13. (4 points) For all $n \geq 1$, find a graph G_n with n vertices such that $\alpha(G_n) = \Omega(n)$ but the Wei-Caro bound is $O(1)$. Recall the meaning of the big-Oh and big-Omega notation: what you need to do is find constants $c, C > 0$ such that $\alpha(G_n) \geq cn$ but $WC(G_n) \leq C$.

HW 6.14. (6 points) Prove that every graph has a bipartite subgraph of size $\geq m/2$ (i. e., you can delete at most half the edges and get a bipartite subgraph). It is required that you use a method analogous to the Wei-Caro proof. This involves

1. defining a probability space;
2. defining a random variable X such that the value of X is always a lower bound on the maximum size of a bipartite subgraph;
3. proving that $E(X) = m/2$

There are also non-randomized ways of solving this problem but such a solution will not earn you credit.

HW 6.15. (3 points) Count the shortest paths between two opposite corners of the $k \times \ell$ grid. The answer is a very simple expression.

HW 6.16. (3 points) In the d-cube Q_d, count the shortest paths between 00...0 and 11...1. The answer a very simple expression.

Notation 6.17. For two vertices u, v, we write $u \cong v$ if either $u = v$ or $u \sim v$. In this case we say that u and v are *adjacent or equal*.

Definition 6.18. Given graphs $G = (V, E)$ and $H = (W, F)$, the **strong product** of G and H is a graph $G * H$ with the vertex set

$$V(G * H) = V \times W.$$

For two vertices vertices (v_1, w_1) and (v_2, w_2), we define $(v_1, w_1) \cong (v_2, w_2)$ if $v_1 \cong v_2$ and $w_1 \cong w_2$.

HW 6.19. (5 points) Consider the graph $C_5 * C_5$ (the 'King's graph' on the 5×5 toroidal grid). Find an independent set of size 5.

BONUS 6.20. (5 points) Show that $\alpha(C_5 * C_5) \leq 5$.

These two problems together assert that $\alpha(C_5 * C_5) = 5$.

DO 6.21. Show that $\alpha(C_7 * C_7) \geq 9$.

BONUS 6.22. (3 points) Show that $\alpha(C_7 * C_7) \leq 10$.

CH+ 6.23. Find $\alpha(C_7 * C_7)$.