Graph Theory: CMSC 27530/37530 Lecture 7

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Revised by instructor

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INDEPENDENT SETS

DO 7.1. Show that $30 \leq \alpha(C_7 * C_7 * C_7) \leq 35$.

CH+ 7.2. (10 points) Determine $\alpha(C_7 * C_7 * C_7)$, or at least narrow the gap between the upper and the lower bounds.

Theorem 7.3 (László Lovász, 1979)**.** $\alpha(\underbrace{C_7 * \ldots * C_7}_{t \text{ copies}}) < w_7^t$ *where* $w_7 = \dfrac{7}{1 + \sec(\pi/7)}$.

The secant function is defined as $\sec(\alpha) = 1/\cos(\alpha)$. This result remains valid if we replace every occurrence of "7" by an odd integer $n \geq 3$.

The bound is not exact–for instance, $w_7^3 \approx 37.1$.

CH++ 7.4. (100 points) Beat the Lovász bound w_7^t on $\alpha(C_7 * \ldots * C_7)$ (t copies) for all values of t in the following sense. Find a number $z < w_7$ such that $\alpha(C_7 * \ldots * C_7) \leq z^t$ for all $t \geq 1$. (Or, prove that no such z exists.)

Either outcome would have significance to Information Theory.

MATCHINGS

Definition 7.5. A **perfect matching** is a matching M which reaches every vertex, i. e., $|M| = n/2$. Note that if a graph has a perfect matching then its order must be even.

HW 7.6 (Kőnig 1916)**. (4 points)** If G is a regular bipartite graph of degree at least 1, then G has a perfect matching. Use either Kőnig's Theorem $\tau = \nu$ or Hall's Marriage Theorem.

Remark 7.7. Note that in 1916, Kőnig was not in the position to use either Kőnig's Theorem (1931) or Hall's Theorem (1935).

DO 7.8. Find a regular graph G of degree at least 1 that does not have a perfect matching. It should satisfy

(a) n is even

(b) n is even and G is connected.

Recall Kőnig's Theorem, which says for that a bipartite graph G, we have $\tau(G) = \nu(G)$. From Kőnig's Theorem we can derive Hall's Theorem: for a bipartite graph G, with $V = L \sqcup R$, if $(\forall A \subseteq L)(|N(A)| \geq |A|)$ then L can be matched, i. e., $\nu = |L|$. We outline one possible proof.

One direction is trivial (if L can be matched, then the Hall conditions are immediate).

Going in the other direction, we assume the Hall conditions hold and from this we infer that $\nu = |L|$. By Kőnig's Theorem, this is equivalent to $\tau = |L|$, which is equivalent to saying $\tau \geq |L|$ (why?), which is the same saying that if C is a cover then $|C| \geq |L|$. So it suffices to prove this last statement.

Let C be a cover, and let $A = C \cap L$, $B = C \cap R$. Let $A' = L \setminus A$. By the Hall condition, $|N(A')| \geq |A'|$. But $N(A') \subseteq B$. As a result,

$$\begin{aligned} |C| &= |A| + |B| \\ &\geq |A| + |N(A')| \\ &\geq |A| + |A'| = |L|. \end{aligned}$$

DO 7.9. (a) If M is a matching and C is a cover, then $|M| \leq |C|$.

(b) If $|M| = |C|$ then both M and C are optimal and therefore $\nu = \tau$.

Proof of Kőnig's Theorem. Let M be a maximal matching obtained by the greedy algorithm. How can we improve M? Say that the edges contained in M are red, and the edges not contained in M are blue. Suppose we can find a path $l_1 r_1 \, l_2 r_2 \ldots l_k r_k$ where the $l_i \in L$, $r_i \in R$, vertices l_1 and r_k (the endpoints of the path) are unmatched, and the edges satisfy the following conditions.

1. $\{l_i, r_i\}$ is blue (not contained in the matching)
2. $\{r_i, l_{i+1}\}$ is red (contained in the matching).

We call a path of this type an *augmenting path*, for the following reason: if we remove all of the red edges on the path from M (there are $k-1$ red edges) and add all of the blue edges on the path to M (there are k blue edges on the path), then the resulting set M' is a matching, and $|M'| = |M| + 1$.

Since we cannot increase M indefinitely, after a finite number of rounds we shall have a matching so that there is no augmenting path. In this case we claim that M is maximal, and we shall demonstrate this by finding a cover C such that $|M| = |C|$. Accoring to DO exercise 7.9 this proves that $\tau = \nu$.

Here is how we construct C. Let W be the set of vertices of M, let $P = W \cap R$ and $Q = W \cap L$. Let A be the set of unmatched vertices of R and let U be the set of vertices reachable from A along alternating paths (a path which uses alternately blue/red edges, necessarily starting with blue since no red edge ends in A). Note that $A \subseteq U$.

DO 7.10. A vertex is reachable from A along an alternating path if and only if it is reachable along an alternating walk.

DO 7.11. $U \cap L = U \cap Q$. (Hint. The reason is that there is no augmenting path (or walk). Explain.)

DO 7.12. $R \setminus U = P \setminus U$. (Hint. The reason is that $A \subseteq U$. Explain.)

Now define the set C by $C := (U \cap Q) \cup (P \setminus U)$. Since C picks one vertex from each edge of M, it follows that $|C| = |M|$. We claim that C is a cover.

If e is a red edge, it is covered by C since we picked one endpoint of each red edge.

Let $e = xy$ be a blue edge where $x \in R$ and $y \in L$.

Case 1. $x \in U$. In this case $y \in U$. Indeed, if x is reachable by an alternating walk and $x \in R$ then the last step in this walk was red, so we can add the xy edge to the walk. But then $y \in U \cap L = U \cap Q$ (by DO exercise 7.11; note that this is the point in the proof where we use the assumption that there is no augmenting path). Therefore $y \in C$, so C covers e.
Case 2. $x \notin U$, so $x \in R \setminus U = P \setminus U$. (Here we used DO exercise 7.12.) But then $x \in C$, so again, C covers e.

This completes the proof that C is a cover, and therefore it completes the proof of Kőnig's Theorem. □

DO! 7.13. Prove that C is a cover without reading the proof above again, so don't read the part after the paragraph where C is defined.

EXTREMAL GRAPH THEORY

Here is an exercise in **extremal graph theory**. It is unusual in that the parameter we fix is not the order but the size of the graph.

CH 7.14. (12 points) For a graph G, let $t(G)$ be the number of triangles. Show that

$$t \leq \frac{\sqrt{2}}{3} m^{3/2}. \tag{1}$$

HW 7.15. (3 points, elegance counts) Prove: for K_n, inequality (1) is asymptotically tight, i.e., $t_n \sim \frac{\sqrt{2}}{3} m_n^{3/2}$, where $t_n = t(K_n)$, $m_n = |E(K_n)|$. (Recall that the "asymptotic equality" notation $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$; see Definition 2.9.)

This exercise shows that Eq. (1) the coefficient $\sqrt{2}/3$ is best possible for infinitely many values of m.

CHROMATIC POLYNOMIAL

Recall the **chromatic polynomial** $f_G(x)$, which counts the legal colorings $g : V \to [x]$, where $x \in \mathbb{N}$.

Theorem 7.16 (George David Birkhoff, 1912)**.** $(\forall G)$(*f_G is a polynomial*).

Proof. Let Π be a partition[1] of V into k independent sets, $A_1, A_2, ...A_k$. Note that every legal coloring of G defines such a partition Π: the blocks of the partition are the sets of

[1] By a partition Π of the set V we mean a set $\Pi = \{A_1, \dots, A_k\}$ where the A_i are subsets of V such that $(\forall i)(A_i \neq \varnothing)$, $(\forall i \neq j)(A_i \cap A_j = \varnothing)$, and $\bigcup_i A_i = V$. The A_i are called the *blocks* of the partition Π.

vertices of equal color. (In other words, these are the equivalence classes of the equivalence relation "vertices u and v have the same color.)

Let $N_\Pi(\text{x})$ be the number of colorings that produce the partition Π. Then $N_\Pi(x) = x(x-1)\dots(x-k+1)$. So this is a polynomial in x. There is only a finite number of such partitions (the number depends on G), and $f_G(x) = \sum_\Pi N_\Pi(x)$. Since every term on the right-hand side is a polynomial in x, so is their sum. □

HW 7.17. (3 points, due Tuesday, April 30) Given a graph, determine the degree and the leading coefficient of the chromatic polynomial of G. (Example: the polynomial $5x^3 + 8x - 32$ has degree 3 and leading coefficient 5. The *leading term* of this polynomial is $5x^3$.)

Definition 7.18. An **orientation** of a graph G is an assignment of a direction to each edge. So a graph has 2^m orientations. An orientation is **acyclic** if no directed cycle arises.

CH 7.19. (7 points) The number of acyclic orientations of a graph G is $\pm f_G(-1)$. (Do NOT look it up!)

DO 7.20. Show that every graph has an acyclic orientation.

HW 7.21. (3+4+1 points, due Tuesday, April 30)

(a) Determine the chromatic polynomials of all trees, complete graphs, and empty graphs (complements of complete graphs).

(b) Count the acyclic orientations of all trees, complete graphs, and empty graphs.

(c) Verify CH 7.19 for these classes of graphs.

In each case you should get simple expressions.

PLEASE SUBMIT ALL HOMEWORK ON ITS DUE DATE. Early submission causes clerical problems for the TA and the instructor; it makes it more difficult for us to keep track of your solution and to properly credit it.