Graph Theory: CMSC 27530/37530 Lecture 8

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EXCLUDED SUBGRAPHS VS. CHROMATIC NUMBER

CH 8.1. (7 points) If $G \not\supset C_5$ then $\chi(G) = O(\sqrt{n})$.

CH 8.2. (10 points) If $G \not\supset K_4$ then $\chi(G) = O(n^{2/3})$.

INDEPENDENCE NUMBER VIA PIGEONHOLE PRINCIPLE

Recall a previous problem, to show that $\alpha(\text{Grid}(k, \ell)) = \lceil n/2 \rceil$ where $n = k\ell$.

There are two parts to this problem.

The easy part is to prove the lower bound $\alpha(\text{Grid}(k, \ell)) \geq \lceil n/2 \rceil$.

This, in fact, holds for all bipartite graphs.

DO 8.3. If G is bipartite graph then $\alpha(G) \geq \lceil n/2 \rceil$.

Proof. 2-color the graph, and pick the largest of the two color-classes. □

In fact, we have already proved a more genaral result: for all graphs G we have

$$\alpha(G) \geq \lceil n/\chi(G) \rceil. \tag{1}$$

So the substance of the proof that $\alpha(\text{Grid}(k, \ell)) = \lceil n/2 \rceil$ is in proving the upper bound

$$\alpha(\text{Grid}(k, \ell)) \leq \lceil n/2 \rceil \tag{2}$$

(where $n = k\ell$).

So let A be an independent set in $\text{Grid}(k, \ell)$. We need to prove that $|A| \leq \lceil n/2 \rceil$. The proof must apply to all possible independent sets, no matter how cleverly designed.

This is an AHA-problem: there is a very short, elegant, convincing argument. When you see it, you slap your forehead and say "AHA." No boring case-distinctions like what happens when we try to describe the structure of the maximum indepedent sets, no tedious calculations, no subtle manipulation of rounding, no hand-waving like "the other cases can be treated similarly."

In fact, we shall give three such proofs.

For a warm-up, let us begin with finding α for paths.

Proposition 8.4. $\alpha(P_n) = \lceil \frac{n}{2} \rceil$.

Proof Prop. 8.4. The lower bound is trivial (DO 8.3). Let us prove the upper bound.

Let us consider the set M consisting of every other edge of the path starting at the left. So $|M| = \lfloor n/2 \rfloor$. ($M$ is a maximum matching in P_n.)

Let A be an independent set in P_n. Then A has at most one vertex from each edge in M. So if n is even then $|A| \le |M| \le n/2$. If n is odd then A might also include the rightmost vertex of the path, so $|A| \le |M| + 1 = (n-1)/2 + 1 = (n+1)/2 = \lceil n/2 \rceil$. □

This was a *pigeonhole argument;* the edges of M plus the rightmost vertex if n is odd are the $\lceil n/2 \rceil$ holes; the pigeons are the elements of the indepedent set A. Each hole can hold at most one pigeon. AHA!

Proof #1 of Inequality (2)*.* We follow the same idea as in the proof of Lemma 8.4. If $n = k\ell$ is even then the grid has a perfect matching, so $\alpha \le n/2$. If n is odd, then removing a corner, the rest has a perfect matching, so $\alpha \le \lceil n/2 \rceil$ just as in the proof of Lemma 8.4. AHA! □

After the second proof, you will slap your forehead even harder. How did I miss *this ?*

Definition 8.5. A subgraph $H \subseteq G$ is a **spanning subgraph** if $V(H) = V(G)$.

DO 8.6. If H is a spanning subgraph of G, then $\alpha(G) \le \alpha(H)$.

Definition 8.7. A spanning path is called a **Hamilton path**.

Proof #2 of Inequality (2)*.* Observe that $\text{Grid}(k, \ell)$ contains a Hamilton path $P_{k\ell} = P_n$ (traverse the grid in a snake-like order). Therefore

$$\alpha(\text{Grid}(k, \ell)) \le \alpha(P_n) = \lceil n/2 \rceil. \tag{3}$$

AHA!! □

Last, we give a conceptualized version of the first proof.

Lemma 8.8. *For all graphs, $\alpha \le n - \nu$.*

Proof. $\alpha = n - \tau \le n - \nu$. □

Proof #3 of Inequality (2)*.* In the grid, $\nu = \lfloor n/2 \rfloor$. So, by Lemma 8.8, $\alpha \le n - \nu = n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$. □

Here's a classic recreational math puzzle.

DO 8.9. Can we tile a chessboard, with two opposite corners removed, with dominoes? (A domino covers two adjacent cells of the chessboard.) Give an AHA proof that this is impossible.

BONUS 8.10. (5 points) For what values of n can we tile the $n \times n$ chess board, with one corner removed, by *triominoes?* A triomino is a 1×3 rectangles placed either horizontally or vertically.

HW 8.11. (4 points) Show that $\alpha(G * \overline{G}) \geq n$. (Here '$*$' indicates the *strong product*; see Definition 6.18.)

INDEPENDENTS SETS OF GRAPHS IN INFORMATION THEORY:
THE SHANNON CAPACITY OF GRAPHS

Suppose an alphabet has five characters. How many strings of length ℓ can be obtained from this alphabet? The answer is 5^ℓ. What if some pairs of characters are easy to confuse? We want to limit our set of possible messages so as to permit only pairwise distinguishable messages.

Let G be a graph whose vertex set is the alphabet. Make two vertices adjacent if the corresponding characters can be confused. How many messages can we now permit? If $\ell = 1$ then this number is $\alpha(G)$. For messages of length ℓ, this number is $\alpha(G^{*\ell})$ where $*$ indicates the strong product and $G^{*\ell}) = \underbrace{G * G * \cdots * G}_{\ell \text{ times}}$. (Why?)

We can measure the information content of our alphabet by the number θ_ℓ such that the number of messages of length ℓ we permit is θ^ℓ. We are interested in large ℓ, we take the limit of θ_ℓ as $\ell \to \infty$. This limit is called the *Shannon capacity* of the graph, introduced by Claude E. Shannon (1916–2001), creator of information theory, in 1956 in a paper titled "The zero-error capacity of a noisy channel."

Definition 8.12. The **Shannon capacity** of a graph, denoted $\Theta(G)$, is defined

$$\Theta(G) = \lim_{\ell \to \infty} \sqrt[\ell]{\alpha(G^{*\ell})}.$$

The '$*$' in the exponent indicates that the power is taken with respect to the strong product.

We shall see later that this limit always exists (Thm. 8.18).

DO 8.13. Show that

(a) $\Theta(G) \geq \alpha(G)$

(b) $\Theta(G) \leq \chi(\overline{G})$.

Corollary 8.14. *If $\alpha(G) = \chi(\overline{G})$, then this number is $\Theta(G)$.*

DO 8.15. If $\overline{G}$ is bipartite, then $\alpha(G) = \chi(\overline{G})$.

BONUS 8.16. (6 points) If G is bipartite, then $\alpha(G) = \chi(\overline{G})$.

The smallest graph G such that neither G nor its complement is bipartite is the pentagon.

The value of $\Theta(C_5)$ was an open problem for more than two decades. The above results show that $2 \leq \Theta(C_5) \leq 3$. Shannon showed (1956) that

$$\sqrt{5} \leq \Theta(C_5) \leq 5/2. \tag{4}$$

For the upper bound, he used Linear Programming. The exact value of $\Theta(C_5)$ remained an open problem for two decades. Finally, in one of the most influential papers in combinatorics over the past half century, László Lovász (1979) introduced powerful linear algebraic techniques to prove upper bounds on the Shannon capacity; in particular, he settled the Shannon capacity of the pentagon: $\Theta(C_5) = \sqrt{5}$.

However, the value of $\Theta(C_7)$ remains open to this day. Lovász proved that for all odd values $n \geq 3$ we have

$$\Theta(C_n) \leq \frac{n}{1 + \sec(\pi/n)}.$$

This bound gives the exact value of $\Theta(C_n)$ for $n = 3$ ($\Theta(C_3) = 1$) and $n = 5$ ($\Theta(C_5) = \sqrt{5}$, see DO 8.17), and it remains the best upper bound known for all odd $n \geq 7$.

DO 8.17. Show that

$$\frac{5}{1 + \sec(\pi/5)} = \sqrt{5}. \tag{5}$$

Hint. Consider a regular pentagon. Prove that the ratio of a diagonal to a side is the golden ratio. Use this fact to calculate $\cos(\pi/5)$.

After this brief history, let us study the math. Given the definition of the Shannon capacity, the first question that arises: does that limit exist?

Theorem 8.18. *The limit* $\lim_{\ell\to\infty} \sqrt[\ell]{G^{*\ell}}$ *exists for all graphs* G *and is equal to* $\sup_\ell \sqrt[\ell]{G^{*\ell}}$ *as well as to* $\limsup_\ell \sqrt[\ell]{G^{*\ell}}$.

To see this, first we need the following useful lemma.

DO 8.19 (Fekete's Lemma)**.** If $\{a_\ell\}_{\ell\in\mathbb{N}}$ is a sequence of positive reals and $a_{k+\ell} \geq a_k \cdot a_\ell$ then

$$\lim_{\ell\to\infty} \sqrt[\ell]{a_\ell} = \sup_\ell \sqrt[\ell]{a_\ell} = \limsup_\ell \sqrt[\ell]{a_\ell}.$$

DO 8.20 (Associativity)**.** Show that

$$G * (H * K) \cong (G * H) * K.$$

It follows from this exercise that the k-th strong power G^{*k} is well-defined: it does not depend on the parenthesisation of the expression $G * G * \cdots * G$. In particular, we obtain $G^{*(k+\ell)} = G^{*k} * G^{*\ell}$.

Next we establish that α is a supermultiplicative function on graphs with respect to the strong product.

Lemma 8.21 (Supermultiplicativity)**.** $\alpha(G * H) \geq \alpha(G) \cdot \alpha(H)$.

Proof. Let A and B be independent sets in G and H, respectively. We claim that $A \times B$ is an independent set in the strong product.

Let $g_1, g_2 \in A$ and $h_1, h_2 \in B$, and suppose $(g_1, h_1) \sim (g_2, h_2)$ in $G * H$. By the definition of the strong product, it must be that $g_1 \cong g_2$ and $h_1 \cong h_2$. Since neither $g_1 \sim g_2$ nor $h_1 \sim h_2$, equality must hold in both cases. As a result, $(g_1, h_1) = (g_2, h_2)$, contradicting the assumption of adjacency. $\square$

Proof of Theorem 8.18. We have $G^{*(k+\ell)} = G^{*k} * G^{*\ell}$. It follows by Lemma 8.21 that $\alpha(G^{*(k+\ell)}) \geq \alpha(G^{*k}) \cdot \alpha(G^{*\ell})$. So Fekete's conditions holds for the sequence $a_\ell = \alpha(G^{*\ell})$. Now Fekete's lemma yields the desied conclusion. $\square$

HW 8.22. (3 points) $(\forall \ell)(\sqrt[\ell]{\alpha(G^{*\ell})} \le \Theta(G))$.

Corollary 8.23. $\Theta(C_5) \ge \sqrt{5}$.

Proof. We have seen that $\alpha(C_5 * C_5) \ge 5$. Now the result follows by HW 8.22 (applied with $\ell = 2$). □

HW 8.24. (3 points) Prove: if G is self-complementary (isomorphic to its complement) then $\Theta(G) \ge \sqrt{n}$.

BONUS 8.25 (Submultiplicativity)**. (4 points)** $\chi(G * H) \le \chi(G) \cdot \chi(H)$.

LINEAR PROGRAMMING

Definition 8.26. A **linear program (LP)** consists of a set of linear inequality constraints and a linear objective function. The **linear programming problem** is to maximize (or minimize) the objective function over the reals subject to the constraints. (The LP specifies whether we seek to maximize or minimize the objective function.) An **integer linear program (ILP)** is the same as an LP except that we are only interested in integral solutions ($x_i \in \mathbb{Z}$). The **optimum value** of the LP/ILP is the optimum value (maximum or minimum) of the objective function under the given constraints.

Definition 8.27 (Independent set LP/ILP)**.** Consider the graph $G = ([n], E)$. Let us associate the real variable x_i with vertex i. Consider the following set of linear constraints:

(1) $(\forall i)(x_i \ge 0)$

(2) $(\forall \text{ clique } C \text{ in } G)(\sum_{i \in C} x_i \le 1)$.

We wish to maximize the objective function $\sum_{i=1}^{n} x_i$ under these constraints.

Note that the constraints imply that $0 \le x_i \le 1$.

DO 8.28. Prove: the optimum value of the ILP given by Def. 8.27, is $\alpha(G)$.

Hint: Note that in this case, the value of each x_i must be 0 or 1. View the vector $(x_1, \dots, x_n)$ as the indicator function (characteristic function) of a subset; then constraints (2) say that this is an independent set, and the objective is to maximize its size.

Definition 8.29 (Fractional independence number)**.** Let us define the **fractional independence number** $\alpha^*(G)$ as the the optimum of LP the given by Def. 8.27.

(So we permit the x_i to take any real values.) We call α^* the **linear relaxation** of α.

DO 8.30. $\alpha(G) \le \alpha^*(G)$.

This is because in the ILP we are maximizing over a smaller set of vectors $(x_1, \dots, x_n)$, namely, only integral vectors, whereas for the LP, we allow the x_i to be real.

Proposition 8.31. $\alpha^*(C_5) = 5/2$

Proof. We have the following LP.

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ x_2 + x_3 &\leq 1 \\ x_3 + x_4 &\leq 1 \\ x_4 + x_5 &\leq 1 \\ x_5 + x_1 &\leq 1 \end{aligned}$$

Our objective function is $\sum_{i=1}^{5} x_i$. Let L denote the optimim value.

First we observe that $L \geq 5/2$ by assigning the value $x_1 = \cdots = x_5 = 1/2$ to the variables.

To show that $L \leq 5/2$, add up all the inequalities. This gives $2(x_1 + \cdots + x_5) \leq 5$. □

Theorem 8.32 (Shannon)**.** $\Theta(G) \leq \alpha^*(G)$. *In particular,* $\Theta(C_5) \leq 5/2$.

The constraints we have described for C_5 can be expressed in matrix form as

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

where the inequality is entry-wise. Likewise, we can express the objective function in matrix form as the matrix product

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

The constraints correspond to the cliques. With every clique C, we associate a new "dual" variable y_C. Define a new linear program as follows. For every column A_i in the constraint matrix, require that $A_i^T Y \geq c_i$, where Y is the column vector whose entries are the variables y_C (C a clique), and c_i is the coefficient of x_i in the objective function. The coefficents of the dual objective function will be the entries on the right-hand size in the LP. This is called the **dual LP**; and the original LP is referred to as the **primal LP**.

The **dual** of the linear program has the constraints

(1) $(\forall \text{ clique } C)(y_C \geq 0)$

(2) $(\forall v \in V)(\sum_{C: v \in C} y_C \geq 1)$.

We wish to minimize the objective function $\sum_C y_C$ under these constraints.

Consider the $\{0,1\}$-solutions: the minimum number of cliques that cover all of V. Since a clique in G is an independent set in $\overline{G}$, this number is the same as the minimum number of independent sets of $\overline{G}$ that cover all of V.

DO 8.33. Show that the minimum number of independents sets in the graph G which cover all of V is $\chi(G)$.

So the optimum of the dual ILP above is $\chi(\overline{G})$.

Definition 8.34. The **fractional chromatic number** of the graph G, denoted $\chi^*(G)$, is the optimum of the above dual LP, applied to the complement $\overline{G}$ (so the dual variables are labeled by independent sets rather than cliques of G).

HW 8.35. (3 points) Prove: $\chi(G) \geq \chi^*(G)$.

HW 8.36. (5 points) Show that $\alpha^*(G) \leq \chi^*(\overline{G})$. Prove this directly from the definitions.

HW 8.37. (4 points) Determine the fractional chromatic number of C_5 directly from the definition. (Do not use Cor. 8.38.)

The **Duality Theorem for Linear Programming** asserts that if both the primal and the dual are feasible (the constraints can be satisfied) then their optima are equal.

Corollary 8.38. $\alpha^*(G) = \chi^*(\overline{G})$.

So we have $\alpha(G) \leq \alpha^*(G) = \chi^*(\overline{G}) \leq \chi(\overline{G})$.