# Graph Theory: CMSC 27530/37530 Lecture 9

Lecture by László Babai
Notes by Geoffrey West
Revised by instructor

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ASYMPTOTICS

**HW 9.1. (4 points)** $n! > (n/\mathrm{e})^n$. The proof is one line using a basic fact about the $\mathrm{e}^x$ function. (Do not use Stirling's formula.)

This inequality will help solve a previous exercise: find infinitely many graphs $G$ that have at least $100^n$ longest paths. $K_n$ has $n!/2$ longest paths. We need to show that for sufficently large $n$ we have $n!/2 \geq 100^n$. Indeed, if $n \geq 101\mathrm{e}$ then $n!/2 \geq 101^{n+1}/2 > 100^{n+1}/2 > 100^n$.

A previous homework problem was to show $t_n \sim \frac{\sqrt{2}}{3} m_n^{3/2}$ where $t_n$ is the number of triangles in $K_n$ and $m_n$ is the number of edges in $K_n$.

**DO 9.2.** Show that a polynomial is asymptotically equal to its leading term.

Using this result, we know that $\binom{n}{3} \sim n^3/6$.

**DO 9.3.** For all fixed $k$ we have $\binom{n}{k} \sim n^k/(k!)$.

**DO 9.4.** Show that if $a_n \sim b_n$ and $c_n \sim d_n$ then $a_n \cdot c_n \sim b_n \cdot c_n$ and $\dfrac{a_n}{c_n} \sim \dfrac{b_n}{d_n}$.

**DO 9.5.** Show that $\sim$ is an equivalence relation among sequences $\{a_n\}$ having $a_n \neq 0$ for all sufficiently large $n$.

**DO 9.6.** If $a_n \sim b_n$ then $a_n^k \sim b_n^k$ for fixed $k \in \mathbb{R}$, assuming $a_n, b_n > 0$ in case $k$ is not an integer.

We may now undertake the problem armed with new technology.

$$t_n = \binom{n}{3} \sim \frac{n^3}{6}$$

$$m_n = \binom{n}{2} \sim \frac{n^2}{2}$$

$$t_n \sim \frac{n^3}{6} = \frac{\sqrt{2}}{3} \cdot \frac{n^3}{2^{3/2}} \sim \frac{\sqrt{2}}{3} m_n^{3/2}.$$

**Definition 9.7.** For a graph $G = (V, E)$, the **contraction** of an edge $e = \{i, j\}$ is the graph which is identical to $G$ except that $i$ and $j$ are "merged". This graph is denoted $G/e$.

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CHROMATIC POLYNOMIAL

For a graph $G = (V, E)$, let $f_G(x)$ be the number of legal colorings $c : V \to [x]$, where $x \in \mathbb{N}$.

**DO 9.8** (Contraction–deletion recurrence)**.** For an edge $e = \{i, j\}$, we have the recurrence relation

$$f_G(x) = f_{G-e}(x) - f_{G/e}(x). \tag{1}$$

*Proof.*

$$\begin{aligned} f_G(x) &= f_{G-e}(x) - \big(\#\text{ of legal colorings of } G - e \text{ where } c(i) = c(j)\big) \\ &= f_{G-e}(x) - f_{G/e}(x). \end{aligned}$$

□

A second proof that $f_G$ is a polynomial follows immmediately by induction on $m$. Base case: $m = 0$. In this case, $f_{\overline{K_n}} = x^n$ is a polynomial. Next, the contraction-deletion recurrence gives us the inductive step.

**DO 9.9.** If $G$ is planar, then $G - e$ and $G/e$ are planar.

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INDEPENDENCE NUMBER, STRONG PRODUCT, SHANNON CAPACITY

Let us revisit the result that $\alpha(C_5 * C_5) \leq 5$. By finding an independent set of size 5 (by "knight's moves"), we can show that 5 is a lower bound to the independence number. To prove the upper bound, no single example is sufficient, so we require a little theorem that simultaneously handles all independent sets.

One strategy is the "averaging argument". For an independent set $S$ in $C_5$, let $x_i \in \{0, 1\}$ be defined

$$x_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S. \end{cases}$$

The following inequalities always hold:

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ x_2 + x_3 &\leq 1 \\ x_3 + x_4 &\leq 1 \\ x_4 + x_5 &\leq 1 \\ x_5 + x_1 &\leq 1. \end{aligned}$$

By taking the sum and dividing by 2 we have $|S| = \sum_{i=1}^{5} x_i \leq \frac{5}{2}$.

**DO 9.10.** For any $G$, $\alpha(K_2 * G) = \alpha(G)$.

We return again to the case of $C_5 * C_5$. Pick an independent set $S$; let $S_i = S \cap \{i\text{th column}\}$, and let $y_i = |S_i|$. It follows from the previous exercise that for any pair of adjacent columns, $k$ and $k+1 \pmod 5$, we have $y_k + y_{k+1} \leq \alpha(C_5) = 2$. Using the averaging approach, we have the inequalities

$$\begin{aligned} y_1 + y_2 &\leq 2 \\ y_2 + y_3 &\leq 2 \\ y_3 + y_4 &\leq 2 \\ y_4 + y_5 &\leq 2 \\ y_5 + y_1 &\leq 2. \end{aligned}$$

Taking the sum and dividing by 2, we conlcude that $\sum_{i=1}^{5} y_i \leq 5$.

Consider the case of $\alpha(C_7 * C_7 * C_7)$. From the result that $\alpha(C_7 * C_7) = 10$ together with supermultiplicativity, it follows that $\alpha(C_7 * C_7 * C_7) \geq 30$. By the averaging technique, we can obtain the result that $\alpha(C_7 * C_7 * C_7) \leq 35$.

**CH 9.11.** Prove that $\alpha(C_7 * C_7 * C_7) \leq 33$.

One of your classmates has shown $\alpha(C_7 * C_7 * C_7) = 33$. Improving the lower bound $\alpha(C_7 * C_7 * C_7) \geq 30$ is no longer assigned.

Recall the *Shannon capacity* of $G$, defined by

$$\Theta(G) = \lim_{k \to \infty} \sqrt[k]{\alpha(G^k)} = \sup_k \sqrt[k]{\alpha(G^k)}.$$

(The exponent here corresponds to a strong product.)

The right-hand equality helps us find a lower bound for $\Theta(G)$. Given that $\alpha(C_7 * C_7) = 10$, it follows that

$$\Theta(C_7) \geq \sqrt{\alpha(C_7 * C_7)} = \sqrt{10} \approx 3.16.$$

Given that $\alpha(C_7 * C_7 * C_7) = 33$, we can improve this bound to $\Theta(C_7) \geq \sqrt[3]{33} \approx 3.21$. In 2017, Ashik Mathew and Patrick Östergøard published the inequality $\alpha(C_7^5) \geq 350$, thus further improving the Shannon capacity bound to $\Theta(C_7) \geq \sqrt[5]{350} \approx 3.227$.

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LINEAR PROGRAMMING

A linear programming problem consists of a set of linear constraints

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &\leq b_1 \\ \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &\leq b_n \end{aligned}$$

$$x_i \geq 0, \quad 1 \leq i \leq n$$

and a linear objective function

$$c_1x_1 + \ldots + c_nx_n$$

which we seek to maximize. A *feasible solution* is a solution $(x_i)_{i=1}^n$ which satisfies the linear constraints. A linear program is *feasible* if there exists a feasible solution. If an LP is infeasible, we say the maximum to the objective function is $-\infty$.

We can express a linear program using matrix notation.

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

The *transpose* of a matrix is its reflection across the diagonal, so that $(a_{ij}^T) = (a_{ji})$.

**DO 9.12.** $(A^T)^T = A$.

**DO 9.13.** $(AB)^T = B^T A^T$.

**Definition 9.14.** For vectors $\mathbf{x} = (x_1, \dots, x_n)^T$ and $\mathbf{y} = (y_1, \dots, y_n)^T$ over the reals we write $\mathbf{x} \leq \mathbf{y}$ if $(\forall i)(x_i \leq y_i)$. This is a partial order on the vectors.

The linear program now takes the form

$$\max \leftarrow \mathbf{c}^T \mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

The dual LP can be expressed similarly as

$$\min \leftarrow \mathbf{b}^T \mathbf{y} \text{ subject to } A^T \mathbf{y} \geq \mathbf{c},\ \mathbf{y} \geq \mathbf{0}.$$

**Definition 9.15.** A vector $\mathbf{x}$ is a **feasible solution** if it satisfies the constraints

$$A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0.$$

**DO 9.16.** If $\mathbf{v}_1 \leq \mathbf{v}_2$ are vectors and $\mathbf{x} \geq 0$, then it follows that

$$\mathbf{v}_1^T \mathbf{x} \leq \mathbf{v}_2^T \mathbf{x}.$$

**DO 9.17.** If $\mathbf{x}$ is a feasible solutions of the primal LP and $\mathbf{y}$ is a feasible solutions to the dual LP then

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}.$$

*Proof.* Using exercise DO 9.16, from the constraints

(i) $A\mathbf{x} \leq \mathbf{b}$

(ii) $A^T \mathbf{y} \geq \mathbf{c}$

(iii) $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$

we deduce that

$$\mathbf{c}^T \mathbf{x} \leq (A^T \mathbf{y})^T \mathbf{x} = \mathbf{y}^T A \mathbf{x} \leq \mathbf{y}^T \mathbf{b} = (y^T \mathbf{b})^T = \mathbf{b}^T \mathbf{y}.$$

The equality $\mathbf{y}^T \mathbf{b} = (y^T \mathbf{b})^T$ hold because the matrix $\mathbf{y}^T \mathbf{b}$ is $1 \times 1$, so it is equal to its transpose. □

**Theorem 9.18** (LP Duality)**.** *If both the primal and the dual are feasible, then the maximum of the primal is equal to the minimum of the dual.*

The LP Duality theorem gives us a "good characterization" result about a linear program. To prove that a solution to the primal is maximum, we only need to exhibit a solution to the dual giving the same value.

**DO 9.19.** It follows from LP Duality that $\alpha^*(G) = \chi^*(\overline{G})$.

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ESTIMATING THE SHANNON CAPACITY.
ORTHONORMAL REPRESENTATION OF GRAPHS

Shannon showed that

$$\alpha(G) \le \Theta(G) \le \alpha^*(G). \tag{2}$$

Using the previous exercise, we deduce that

$$\alpha(G) \le \Theta(G) \le \alpha^*(G) = \chi^*(\overline{G}) \le \chi(\overline{G}).$$

We have seen that the independence number is supermultiplicative:

$$\alpha(G * H) \ge \alpha(G) \cdot \alpha(H).$$

**DO 9.20.** $\chi(\overline{G * H}) \le \chi(\overline{G}) \cdot \chi(\overline{H})$.

**DO 9.21.** (a) $\overline{G} * \overline{H} \subseteq \overline{G * H}$

(b) If $\overline{G} * \overline{H} = \overline{G * H}$ then either one of the graphs $G$, $H$ has just one vertex, or both graphs are empty, or both graphs are complete.

**Lemma 9.22.** *If $f : \{Graphs\} \to \mathbb{R}$ is such that*

*(i)* $(\forall G)(\alpha(G) \le f(G))$

*(ii) the function $f$ is submultiplicative:* $(\forall G, H)(f(G * H) \le f(G) \cdot f(H))$

*then* $(\forall G)(\Theta(G) \le f(G))$.

**HW 9.23. (5 points)** Prove the lemma.

**Corollary 9.24.** $\Theta(G) \le \chi(\overline{G})$.

**DO 9.25.** $\chi^*(\overline{G})$ is submultiplicative.

**Corollary 9.26** (Shannon)**.** $\Theta(G) \le \chi^*(\overline{G}) = \alpha^*(G)$.

Recall the exercise that $\alpha(G * \overline{G}) \ge n$. The proof is simple: $S = \{(x, x) \mid x \in V\}$ is an indepedent set. It follows from this result that for self-complementary graphs, $\alpha(G^2) \ge n$ and therefore $\Theta(G) \ge \sqrt{n}$, solving another exercise.
Using the fact that $C_5$ is self-complementary, it follows that $\Theta(C_5) \ge \sqrt{5}$. Lovász proved that this is the exact value of $\Theta(C_5)$.

**Definition 9.27.** The **norm** of a vector $\mathbf{x} \in \mathbb{R}^d$ is $\sqrt{\mathbf{x}^T\mathbf{x}}$. We denote this value by $\|\mathbf{x}\|$.

**Definition 9.28.** The standard **dot product** in $\mathbb{R}^d$ is $\mathbf{x}^T\mathbf{y} = \sum_{i=1}^d x_i y_i$, denoted by $\mathbf{x} \cdot \mathbf{y}$.

**Definition 9.29.** Two vectors $\mathbf{v}_1, \mathbf{v}_2$ are **orthogonal** if $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. In this case we write $\mathbf{v}_1 \perp \mathbf{v}_2$.

**Definition 9.30** (Lovász)**.** An **orthonormal representation** (ONR) of a graph $G = (V, E)$ in dimension $d$ is a collection of vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{R}^d$ satisfying

(i) $(\forall i)(\|\mathbf{v}_i\| = 1)$

(ii) $(\forall i \not\simeq j)(\mathbf{v}_i \perp \mathbf{v}_j)$.

**Definition 9.31.** The **Lovász dimension** of a graph $G$ is the minimum dimension $d$ such that the graph has an ONR in $\mathbb{R}^d$. We denote this number L-dim$(G)$.

**DO 9.32.** L-dim$(G) = 1$ if and only if $G$ is complete.

**HW 9.33. (5 points)** L-dim$(G) \leq \chi(\overline{G})$ .

Last night's version of this problem set erroneously claimed that there exists a non-bipartite graph with L-dim= 2. In fact, no such graph exists, so here is the revised version of the problem. Thanks to Shashank for pointing out my error.

**DO 9.34** (Updated May 1, 1pm)**.** Show that L-dim$(G) \leq 2$ if and only if $\overline{G}$ is bipartite.

**HW 9.35. (3 points)** L-dim$(C_5) = 3$.

**HW 9.36. (2+2 points)** True or false:

(a) For all graphs $G$, L-dim$(G) \leq \chi^*(G)$.

(b) For all graphs $G$, L-dim$(G) \geq \alpha^*(G)$.

If true, prove; if false, give a counterexample and reason why it is a counterexample.

The following three problems were stated in class and in last night's version of this sheet as HW for Tuesday. I am downgrading their status (May 1, 1pm); I will discuss them in Thursday's class. Please solve the two DO exercises among them before Thursday's class.

**DO 9.37.** L-dim$(G) \geq \alpha(G)$.

**Exercise 9.38.** The L-dim function is submultiplicative.

**DO 9.39.** Infer $\Theta(G) \leq$ L-dim$(G)$ from the preceding two problems.