Graph Theory: CMSC 27530/37530 Lecture 10

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PERFECT GRAPHS, POSETS, COMPARABILITY GRAPHS

Recall that $H = (W, F)$ is a *subgraph* of $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$; we write $H \subseteq G$. If $W = V$, then H is a *spanning subgraph*.

DO 10.1. The number of spanning subgraphs is 2^m.

Definition 10.2. $H = (W, F)$ is an **induced subgraph** of $G = (V, E)$ if $W \subseteq V$ and $F = E \cap \binom{W}{2}$. We denote this subgraph $G[W]$.

DO 10.3. The number of induced subgraphs is 2^n.

Recall the clique number: $\omega(G) = \alpha(\overline{G})$ is the size of the largest clique in G.

If $H \subseteq G$, then $\omega(H) \leq \omega(G)$. What can be said about $\alpha(H)$ compared to $\alpha(G)$? Nothing can be said for subgraphs is general.

DO 10.4. If H is an induced subgraph of G then $\alpha(H) \leq \alpha(G)$.

DO 10.5. If H is a spanning subgraph of G then $\alpha(H) \geq \alpha(G)$.

So α is monotone increasing with respect to induced subgraphs; and monotone decreasing with respect to spanning subgraphs. ω is monotone increasing with respect to all subgraphs.

For all graphs G we have

$$\chi(G) \geq \omega(G). \tag{1}$$

DO 10.6. If G is bipartite then $\chi(G) = \omega(G)$.

Proof. If G is bipartite, not empty, then $\chi(G) = \omega(G) = 2$. If G is empty, then both numbers are equal to 1. □

Definition 10.7. A graph G is **perfect** if $\chi(G') = \omega(G')$ for all induced subgraphs $G' \subseteq G$.

DO 10.8. All bipartite graphs are perfect.

DO 10.9. For what values of n is C_n perfect?

Definition 10.10. A **graph property** $\mathcal{P}$ is an isomorphism invariant predicate on graphs. Being a *predicate* means $\mathcal{P}$ is a function from all graphs to $\{0,1\}$; 0 means "false," 1 means "true." If $\mathcal{P}(G) = 1$ then we say that graph G has proeprty $\mathcal{P}$. Isomorphism invariance means if $G \cong H$ then $\mathcal{P}(G) = \mathcal{P}(H)$. Examples: being bipartite; more generally, being k-colorable for a given k; being triangle-free; being planar; being connected; etc. etc.

Definition 10.11. We say that a graph property $\mathcal{P}$ is **hereditary to subgraphs** if whenever a graph G has property $\mathcal{P}$, all its subgraphs have the property.

DO 10.12. The following properties are hereditary to subgraphs: being bipartite; more generally, for being k-colorable for a given k; planarity; being triangle-free.

Definition 10.13. We say that a graph property $\mathcal{P}$ is **hereditary to induced subgraphs** if whenever a graph G has property $\mathcal{P}$, all its induced subgraphs have the property.

DO 10.14. The property that $\overline{G}$ is bipartite is not hereditary to subgraphs but is hereditary to induced subgraphs. More generally, for every k, the property that $\overline{G}$ is k-colorable is not hereditary to subgraphs but is hereditary to induced subgraphs.

The following exercise is a lemma to the preceding exercise.

DO 10.15. $\overline{G}[A] = \overline{G[A]}$.

Theorem 10.16. *If $\overline{G}$ is bipartite then G is perfect.*

In the light of exercise DO 10.14, we only need to prove that if $\overline{G}$ is bipartite then $\chi(G) = \omega(G)$. Switching to the complement of G, this is equivalent to the following.

Claim 10.17. If G is bipartite then $\chi(\overline{G}) = \alpha(G)$.

We start with a lemma. Your proof should be 2 lines.

HW 10.18. (4 points) $(\forall G)(\chi(\overline{G}) \le n - \nu(G))$.

Proof of Theorem 10.16. We prove Claim 10.17. For all graphs we have

$$\chi(\overline{G}) \ge \alpha(G) = \omega(\overline{G}).$$

We need to show the converse inequality under the assumption that G is bipartite. Indeed,

$$\alpha(G) = n - \tau(G) = n - \nu(G) \ge \chi(\overline{G})$$

where the leftmost equation holds for all graphs by definition, the rightmost inequality holds for all graphs by the lemma (HW 10.18), and the middle equation is Kőnig's Theorem. $\square$

HW 10.19. (6 points) For what values of $n \ge 3$ is $\overline{C}_n$ perfect?

Definition 10.20. A **partially ordered set** or **poset** is a pair $(S, \le)$, where S is a set and $\le$ is a relation on the set S that is reflexive, transitive, and antisymmetric. Antisymmetry means

$$(x \le y \ \wedge \ y \le x) \Rightarrow x = y.$$

We write $x < y$ to express that $x \le y$ and $x \ne y$.

($\wedge$ denotes "AND.")

Example 10.21. Let Ω be a set and $\mathcal{P}(\Omega)$ its powerset, i. e., the set of all subsets of Ω. Then $(\mathcal{P}(\Omega), \subseteq)$ is a partially ordered set. We refer to this example as the Boolean poset.

Definition 10.22. Two elements x, y of a poset are **comparable** if $x \leq y$ or $y \leq x$.

DO 10.23. The comparability relation is reflexive and symmetric.

DO 10.24. Give a poset in which the comparability relation is not transitive. Make your example as small as possible.

Definition 10.25. The **comparability graph of a poset** $(S, \leq)$ is the graph having the vertex set S and adjacency relation $a \sim b$ if a and b are comparable.
A graph G is said to be a **comparability graph** if there exists a poset of which G is the comparability graph.

DO 10.26. Prove that comparability graphs are hereditary with respect to induced subgraphs.

Definition 10.27. A **chain** in a poset is a linearly ordered subset, i. e., a set $\{a_1, \dots, a_k\}$ such that $a_1 < a_2 < \dots < a_k$.

DO 10.28. Show that the cliques in the comparability graph of a poset are precisely the chains.

HW 10.29. (4 points) Prove that comparability graphs are perfect.

DO 10.30. Is C_4 a comparability graph?

The answer is yes. The following sets, ordered by inclusion, provide one illustration.

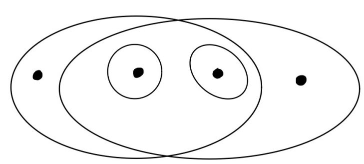

Figure 1: A poset with C_4 comparability graph.

DO 10.31. Find the smallest graph that is not a comparability graph. Show that it is C_5. Show that C_5 is not a comparability graph because it is not perfect. Show that all graphs with ≤ 4 vertices are comparability graphs.

DO 10.32. Find the smallest graph that is not perfect. (C_5)

BONUS 10.33. (4 points) Prove: every bipartite graph is a comparability graph.

CH 10.34. (6 points) Is there a perfect graph that is not a comparability graph?

DO 10.35. If G is perfect then $\Theta(G) = \alpha(G)$.

The following is a central result in the combinatorial theory of posets.

Definition 10.36. In a poset, an **antichain** is a set of pairwise incomparable elements.

Theorem 10.37 (Dilworth)**.** *Given a poset $(S, \leq)$, the maximum size of an antichain is equal to the minimum number of chains of which S is the union. (Such a set of chains is called a* chain cover*.)*

DO 10.38. Prove the trivial direction of the theorem: max $\leq$ min.

The proof of the trivial direction is based on the following observation.

DO 10.39. If C is a chain and A is an antichain then $|C \cap A| \leq 1$.

Definition 10.40. For a graph $G = (V, E)$,

$$\rho(G) = \text{minimum } \# \text{ of elements of } V \cup E \text{ whose union is } V.$$

HW 10.41. (4 points) Prove: $(\forall G)(\rho + \nu = n)$. Your proof should be not more than four lines.

BONUS 10.42. (4 points) Deduce Kőnig from Dilworth in 4 lines, using HW 10.41.

Definition 10.43. For a poset $(S, \leq)$, the **incomparability graph** is the complement of the comparability graph.

DO 10.44. Dilworth's theorem is equivalent to saying that the incomparability graph is perfect.

In the early 1960s, Claude Berge (1926–2002) made the following conjecture, now known as the **Perfect Graph Theorem**.

Theorem 10.45 (Perfect Graph Theorem, Lovász, 1972)**.** *The complement of a perfect graph is perfect.*

Remark 10.46. Note that the Perfect Graph Theorem, combined with the trivial observation that bipartite graphs are perfect, yields the nontrivial result that the complement of a bipartite graph is perfect. Note that this consequence is essentially Kőnig's Theorem.

HW 10.47. (4 points) Use the Perfect Graph Theorem to prove Dilworth's Theorem.

IMPOSSIBILITY OF TILING VIA INVARIANTS

DO 10.48. Show that the 8×8 chessboard with two opposite corners removed cannot be tiled by dominoes. (A domino covers two adjacent cells of the board.)

An AHA proof. Color the cells of the board in a chessboard (checkerboard) fashion. Observe that every domino covers one balck and one white cells. This observation yields the following **invariant**, true for every tileable region on the chessboard. By "region" we mean any set of cells of the chessboard.

Tiling invariant #1. In every region tileable by dominoes, the colors are balanced (each color occurs the same number of times)

Let us refer to the region under consideration (the chessboard with two opposite corners removed) as the *truncated board.* Noting that colors on the full chessboard are balanced (because it is tileable), we see that they are not balanced on the truncated board (because we removed two cells of the same color). So the truncated board cannot be tiled. □

DO 10.49. So for tileability of a region on the chessboard, it is necessary that the two colors be balanced. Is it sufficient?

Proof by weights. A particularly elegant way of phrasing the preceding proof is the following. Let us assign weight 1 to black cells and weight -1 to white cells. The weight of a region is the sum of the weights the cells in the region. So the weight of every domino is zero. We obtain the following invariant:

Tiling invariant #2 The weight of every tileable region is zero.

Then we note that the weight of the full chessboard is zero (because it is tileable), so the weight of the truncated board is not zero (since we removed two cells of the same nonzero weight). This again proves that the truncated board cannot be tiled. □

DO 10.50. A **triomino** consists of three adjacent cells in a row or in a column. Consider the $n \times n$ board with one corner removed; call this the *truncated board* for this problem. For what values of n is it possible to tile the truncated board with triominoes?

Solution. One obvious invariant is that the number of cells in any tileable region is divisible by 3. This rules out the numbers n that are divisible by 3, i. e., $n \equiv 0 \pmod 3$. If $n \equiv 1$ mod 3, then it is easy to tile the truncated board.

Claim 10.51. If $n \equiv 2 \pmod 3$ then the truncated board cannot be tiled.

Proof. Use three colors, $0, 1, 2$. Color the cell (i, j) by $i - j \bmod 3$. So the first row will be colored $021021021\ldots$, the second $102102102\ldots$, the third $210210210\ldots$ etc. Now we notice that every triomino covers exactly one cell of each color, so Tiling invariant #1 holds verbatim.

So the question is, are the colors on the truncated board balanced? Depends on which corner we removed. If we removed the top left or the bottom right corner, they are. But if we removed the top right corner, they are not. An easy way to see this, without actually counting, is the following. In the first $n-2$ columns the colors are balanced because $n-2$ is divisible by 3, so this $n \times (n-2)$ region is tileable by horizontally placed triominoes. From the remaining $n \times 2$ region, the last $(n-2)$ rows are balanced because they are tileable by vertically placed triominoes. Finally we are left with the 2×2 upper right corner with the top right corner removed. This is not balanced because its two diagonal cells have the same color. (Notice that the colors are the same along every line parallel to the main diagonal.) $\square$

DO 10.52. What if we removed the wrong corner?

We can also view our colors as weights. In this case we note that the weight of each triomino is zero modulo 3 (this is the analogue of Tiling invariant #2). We see that if the top right corner is removed, the weight of the truncated board is not zero. (As above, the weight of everything but the top right 2×2 corner is zero.) $\square$

Let us now consider "not necessarily contiguous triominoes," meaning any three cells placed in a row or in a column. For emphasis we shall use the term "contiguous triominoes" for the kind of triominoes we have discussed above. The following result was discovered by two of your classmates; they gave clever but not quite "AHA" proofs.

BONUS 10.53. (4 points) Prove: for $n \equiv 2 \pmod 3$, the truncated board cannot be tiled by contiguous horizontal triomonoes and not necessarily contiguous vertical triominoes. Give an AHA proof via invariants.

DO 10.54. If neither the horizontal nor the vertical triominoes need to be contiguous then one can tile the truncated $n \times n$ board when $n \equiv 2 \pmod 3$, $n \geq 5$. In fact, permitting a single non-contiguous horizontal triomino and a single non-contiguous vertical triomino suffices (while all other tiles are contiguous triominoes).

BONUS 10.55. (6 points) Problem 7 ("Band-aid problem") on the instructor's Puzzle Problems sheet, http://people.cs.uchicago.edu/~laci/REU12/puzzles.pdf. (Hover and click.) Do not look up or discuss.

CH 10.56. (9 points) Problem 2 ("Dividing a rectangle") on the instructor's Puzzle Problems sheet. Give an AHA proof. Do not look up or discuss.

KRONECKER PRODUCT, LOVÁSZ DIMENSION, AND SHANNON CAPACITY

Definition 10.57. Let $\mathbf{v} \in \mathbb{R}^k$, $\mathbf{w} \in \mathbb{R}^n$. The **Kronecker product** of $\mathbf{v}$ and $\mathbf{w}$ is the vector $\mathbf{v} \otimes \mathbf{w} \in \mathbb{R}^{kn}$ whose entries are the products $v_i w_j$ ordered by lexicographic order:
$(v_1w_1, v_1w_2, \dots, v_1w_n, v_2w_1, v_2w_2, \dots, v_2w_n, \dots, v_kw_1, v_kw_2, \dots, v_kw_n)^T$ where the v_i are the coordinates of $\mathbf{v}$ and the w_j are the coordinates of $\mathbf{w}$. (Recall that we write vectors as columns, hence the need for the transpose.).

Recall that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$, the *standard dot product* is $\mathbf{a}^T\mathbf{b} = \sum a_i b_i$.

HW 10.58. (3 points) $(\mathbf{a} \otimes \mathbf{x})^T(\mathbf{b} \otimes \mathbf{y}) = (\mathbf{a}^T\mathbf{b})(\mathbf{x}^T\mathbf{y})$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Let $G = (V, E)$ where $V = \{v_1, \dots, v_n\}$. Lovász's orthonormal representation (ONR) of a graph $G = (V, E)$ is a list of vectors, $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{R}^d$, such that

(i) $\mathbf{v}_i^T\mathbf{v}_i = 1$

(ii) $(\forall i, j)($ if $v_i \not\simeq v_j$ then $\mathbf{v}_i^T\mathbf{v}_j = 0)$

HW 10.59. (4 points) If $(\mathbf{v}_1, ..., \mathbf{v}_r)$ is an ONR of G and $(\mathbf{w}_1, ..., \mathbf{w}_s)$ is an ONR of H, then

$$(\mathbf{v}_i \otimes \mathbf{w}_j \mid 1 \le i \le r, 1 \le j \le s)$$

is an ONR of $G * H$ (strong product). (Note that G has r vertices and H has s vertices.)

HW 10.60. (3 points) Prove that the Lovász dimension function L-dim is submultiplicative:

$$\text{L-dim}(G * H) \le \text{L-dim}(G) \cdot \text{L-dim}(H). \tag{2}$$

HW 10.61. (4 points) Prove: $\text{L-dim}(G) \ge \alpha(G)$.

HW 10.62. (3 points) Prove: $\Theta(G) \le \text{L-dim}(G)$.

CH 10.63. (20+5 points) (1) Find a graph such that $\text{L-dim}(G) < \chi(\overline{G})$.
(2) Find a graph such that $\text{L-dim}(G) < \chi^*(\overline{G})$.

Recall that $\text{L-dim}(G) \le \chi(\overline{G})$. The following inequality goes in the opposite direction.

BONUS 10.64. (5 points) Prove: $\chi(\overline{G}) \le 2^{\text{L-dim}(G)}$.

DO 10.65. Let $n \ge 5$ be odd. Prove: $\chi(\overline{C}_n) = (n+1)/2$. (Recall HW 10.18.)

Let $n \ge 5$ be odd. We know that $\alpha(C_n) = (n-1)/2$ and $\chi(\overline{C}_n) = (n+1)/2$, so $(n-1)/2 \le \text{L-dim}(C_n) \le (n+1)/2$.

CH 10.66. (6 points) Let $n \ge 5$ be odd. Prove: $\text{L-dim}(C_n) = (n+1)/2$.