Graph Theory: CMSC 27530/37530 Lecture 11

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May 7, 2019

THE PENTAGON AND THE GOLDEN RATIO

If a and b satisfy that $\frac{a}{b} = \frac{b}{a-b}$, then we say this ratio is the **golden ratio**, denoted by the symbol φ (`\varphi`). We have

$$\varphi = \frac{a}{b} = \frac{b}{a-b} = \frac{1}{\frac{a}{b}-1} = \frac{1}{\varphi - 1}$$

So $\varphi^2 - \varphi = 1$. The positive solution of this quadratic equation is $\varphi = (1+\sqrt{5})/2 \approx 1.618$.

One place the golden ratio arises is the regular pentagon, shown in Figure 1.

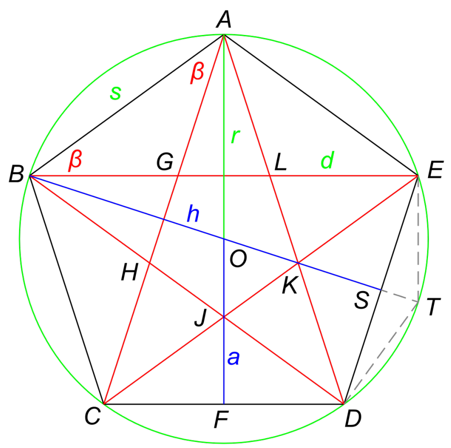

Figure 1: The regular pentagon. Image: Wikimedia Commons *Regular Pentagon Geometry*

DO 11.1. Prove that the ratio of the diagonal of the regular pentagon to its side is the golden ratio: $\dfrac{|BE|}{|BA|} = \varphi$. Use similar triangles in Figure 1.

DO 11.2. $\angle(ABE) = \pi/5$.

DO 11.3. Infer from the previous two exercises that
$\cos(\pi/5) = \varphi/2 = (\sqrt{5}+1)/4 = 1/(\sqrt{5}-1)$.

ORTHONORMAL SYSTEMS

Definition 11.4. An **orthonormal system** $\mathbf{v}_1, \ldots, V_k \in \mathbb{R}^n$ is a set of pairwise orthogonal unit vectors, i. e., $\mathbf{v}_i^T \mathbf{v}_j = \delta_{ij}$. An orthonormal basis (ONB) is an orthonormal system that is a basis.

Here δ_{ij} is the *Kronecker delta notation:*

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

DO 11.5. Every orthonormal system of vectors is linearly independent.

DO 11.6. Every orthonormal system in $\mathbb{R}^n$ can be extended to an orthonormal basis.

(Use Gram-Schmidt orthogonalization.)

Let $\mathbf{b}_1, ..., \mathbf{b}_n$ be a basis of $\mathbb{R}^n$. This means that every vector $\mathbf{v} \in \mathbb{R}^n$ can be uniquely expressed as a linear combination

$$\mathbf{v} = \sum_{j=1}^{n} \beta_j \mathbf{b}_j$$

where the coefficients $\beta_j \in \mathbb{R}$ are called the **coordinates** of $\mathbf{v}$ with respect to the basis $\mathbf{b}_1, ..., \mathbf{b}_n$. To compute the coordinates is usually a tedious job; we need to solve a system of linear equations. However, if the basis is orthonormal, the task reduces to computing dot products. With respect to an ONB, the coordinates are called the "Fourier coefficients" of the vector $\mathbf{v}$.

DO 11.7. Let $\mathbf{b}_1, ..., \mathbf{b}_n$ be an ONB of $\mathbb{R}^n$ and $v = \sum_{i=1}^{n} \beta_i \mathbf{b}_i$ a vector in $\mathbb{R}^n$. Then

$$\beta_i = \mathbf{b}_i^T \mathbf{v}. \tag{1}$$

Coordinates with respect to an ONB easily express the dot product.

DO 11.8. Let $\boldsymbol{\beta} = (\beta_1, \ldots \beta_n)^T$ denote the the coordinate vector of $\mathbf{v}$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots \gamma_n)^T$ the coordinate vector of $\mathbf{w}$ with respect to an ONB. Then

$$\mathbf{v}^T \mathbf{w} = \boldsymbol{\beta}^T \boldsymbol{\gamma}. \tag{2}$$

The **Pythagorean identity** follows: if $\mathbf{b}_1, \ldots, \mathbf{b}_n$ is an ONB then

$$\|\mathbf{v}\|^2 = \sum_{i=1}^{n} (\mathbf{b}_i^T \mathbf{v})^2. \tag{3}$$

Proof. Let $\mathbf{v} = \sum_{i=1}^{n} \beta_i \mathbf{b}_i$. Then

$$\|\mathbf{v}\|^2 = \mathbf{v}^T \mathbf{v} = \boldsymbol{\beta}^T \boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} (\mathbf{b}_i^T \mathbf{v})^2.$$

Here we used several of the preceding results. Which ones? □

A corollary that applies to orthonormal systems follows.

DO 11.9 (Parseval's inequality)**.** If $\mathbf{b}_1, \dots, \mathbf{b}_k$ is an orthonormal system then

$$\|\mathbf{v}\|^2 \geq \sum_{i=1}^{k} (\mathbf{b}_i^T \mathbf{v})^2.$$

Proof. Extend the system to an ONB and apply the Pythagorean theorem. □

KRONECKER PRODUCT, LOVÁSZ DIMENSION, SHANNON CAPACITY

Let $\mathbf{a} \in \mathbb{R}^k$ and $\mathbf{x} \in \mathbb{R}^n$ be column vectors. Recall that the *Kronecker product* of $\mathbf{a}$ and $\mathbf{x}$ is the vector $(\mathbf{a} \otimes \mathbf{x})$ having entries $(\mathbf{a} \otimes \mathbf{x})_{(i,j)} = (a_i b_j)$ where the pairs $(i, j) \in [k] \times [n]$ are arranged in lexicographic order.

Lemma 11.10. *For* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$ *and* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*,*

$$(\mathbf{a} \otimes \mathbf{x})^T (\mathbf{b} \otimes \mathbf{y}) = (\mathbf{a}^T \mathbf{b})(\mathbf{x}^T \mathbf{y}).$$

HW 11.11. (4 points) Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be an ONB of $\mathbb{R}^n$ and let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Consider the vector

$$\mathbf{g} = \sum_{i=1}^{n} \alpha_i (\mathbf{e}_i \otimes \mathbf{e}_i) \in \mathbb{R}^{n^2}.$$

Compute $\|\mathbf{g}\|$. Your answer should be a simple expression in terms of the α_i.

Recall that an orthonormal representation (ONR) of a graph $G = (V, E)$ in dimension d is a collection of vectors, $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{R}^d$, satisfying

(i) $(\forall i)(\|\mathbf{v}_i\| = 1)$

(ii) $(\forall i \not\simeq j)(\mathbf{v}_i \perp \mathbf{v}_j)$.

The *Lovász dimension* of a graph G, written L-dim(G), is the minimum dimension of an ONR of G.

Next we prove that the Lovász dimension is submultiplicative with respect to the strong product.

Lemma 11.12. L-dim$(G * H) \leq$ L-dim$(G) \cdot$ L-dim(H).

Proof. Let $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ and $(\mathbf{w}_1, \dots, \mathbf{w}_k)$ be ONRs of minimum dimension for G and H, respectively. To construct an ONR for $G * H$, assign the vector $\mathbf{v}_i \otimes \mathbf{w}_j$ to vertex $(i, j) \in V(G * H)$. For any two vertices $(i, j) \not\simeq (a, b)$, at least one of the following must hold:

1. $i \not\simeq a$

2. $j \not\simeq b$.

From Lemma 11.10 we have

$$(\mathbf{v}_i \otimes \mathbf{w}_j)^T(\mathbf{v}_a \otimes \mathbf{w}_b) = (\mathbf{v}_i^T \mathbf{v}_a)(\mathbf{w}_j^T \mathbf{w}_b).$$

Now the right-hand side is zero since at least one of its terms is zero. It follows that $(\mathbf{v}_i \otimes \mathbf{w}_j)_{(i,j)\in V(G*H)}$ is an ONR of $G * H$ and it has dimension L-dim$(G) \cdot$ L-dim(H). □

Corollary 11.13. L-dim$(G) \geq \alpha(G)$.

Proof. Let $A \subseteq V(G)$ be independent. The the vectors of an ONR corresponding to the vertices in A form an orthonormal system, so $|A| \leq \dim$ by exercise DO 11.5. □

This result combined with the submultiplicativity of the Lovász dimension (Lemma 11.12) gives the following important result.

Theorem 11.14. $\Theta(G) \leq$ L-dim(G).

This follows from our general criterion for upper bounds on the Shannon capacity (HW 9.23):

Theorem 11.15. *If* $f : \{\textit{Graphs}\} \to \mathbb{R}^+$ *and*

(i) $(\forall G)(\alpha(G) \leq f(G))$

(ii) $(\forall G, H)(f(G * H) \leq f(G) \cdot f(H))$

then $(\forall G)(\Theta(G) \leq f(G))$.

LOVÁSZ CAPACITY : THE LOVÁSZ ϑ FUNCTION

Definition 11.16. By a "handle" we mean any unit vector. We think of an ONR as the ribs of an "umbrella." Given an ONR $(\mathbf{a}_1, ..., \mathbf{a}_n)$ and a handle $\mathbf{c}$ in $\mathbb{R}^d$, we define Value$(\cdot, \cdot)$ as

$$\text{Value}\big((\mathbf{a}_1, ..., \mathbf{a}_n), \mathbf{c}\big) = \max_{i=1,...,n} \frac{1}{(\mathbf{c}^T \mathbf{a}_i)^2}.$$

Note that the quantity $\mathbf{c}^T \mathbf{a}_i$ is the cosine of the angle $\angle(\mathbf{c}, \mathbf{a}_i)$ so the definition of the value selects the largest angle between the handle and the ribs.

Definition 11.17. The **Lovász Capacity** of a graph G is defined as the minimum value:

$$\vartheta(G) = \min_{(\mathbf{a}_1,...,\mathbf{a}_n),\mathbf{c}} \text{Value}\big((\mathbf{a}_1, ..., \mathbf{a}_n), \mathbf{c}\big)$$

where the minimum is taken over all possible choices of the ONR $(\mathbf{a}_1, ..., \mathbf{a}_n)$ of G and the handle $\mathbf{c}$.

(The letter ϑ (`\vartheta`) is the Greek lower case "theta.")

So this definition seeks to minimize the largest angle between the handle and the ribs. We now state the central result of Lovász's theory.

Main Theorem 11.18 (Lovász)**.** *For every graph* G *we have* $\Theta(G) \leq \vartheta(G)$.

As with all known upper bounds on the Shannon capacity, the proof will be based on Theorem 11.15, so we need to prove the following two inequalities.

(1) $\vartheta(G) \ge \alpha(G)$

(2) $\vartheta(G * H) \le \vartheta(G) \cdot \vartheta(H)$.

Proof of $\vartheta \ge \alpha$. Pick any ONR $(\mathbf{a}_1, ..., \mathbf{a}_n)$ and handle $\mathbf{c}$. We need to show $\alpha(G) \le \text{Value}(\text{ONR}, \mathbf{c})$. Pick any independent set $|A|$. Then

$$
\begin{aligned}
1 = \|\mathbf{c}\|^2 &\ge \sum_{i \in A} (\mathbf{c}^T \mathbf{a}_i)^2 \\
&\ge |A| \cdot \min_{i \in A} (\mathbf{c}^T \mathbf{a}_i)^2 \\
&\ge |A| \cdot \min_{i=1,\dots,n} (\mathbf{c}^T \mathbf{a}_i)^2 \\
&= |A| \cdot \frac{1}{\text{Value}(\text{ONR}, \mathbf{c})}
\end{aligned}
$$

In the first step we used Parseval's inequality through the observation that $\{\mathbf{a}_i \mid i \in A\}$ is an orthonormal system.

We conclude that $|A| \le \text{Value}(\text{ONR}, \mathbf{c})$. Since this holds for an arbitrary ONR, handle, and independent set, it follows that $\alpha(G) \le \vartheta(G)$. □

We leave the submultiplicativity of the ϑ function as an exercise.

HW 11.19. (5 points) Prove: $\vartheta(G * H) \le \vartheta(G) \cdot \vartheta(H)$.

This exercise will complete the proof of the Main Theorem.

We remark that Lovász also proved the following. We say that a pair (ONR, handle) is *optimal* if its value is $\vartheta(G)$.

Theorem 11.20 (Equal angles). *There exists an optimal (ONR, handle) pair for which all values* $\mathbf{c}^T \mathbf{a}_i$ *are equal (and therefore their common value is* $1/\sqrt{\vartheta(G)}$*)* .

This phenomenon will be illustrated by the "Lovász umbrella" in the next section. Theorem 11.20 will be an ingredient in the dual characterization of the ϑ function, see Cor. 11.29.

THE SHANNON CAPACITY OF C_5: THE LOVÁSZ UMBRELLA

Shannon (1956) proved that $\sqrt{5} \le \Theta(C_5) \le 5/2$. Closing this gap remained a major open problem in Information Theory until Lov'asz's 1979 paper,
"On the Shannon capacity of a graph," IEEE Transactions on Information Theory, Vol. 25 (1979), pp. 1–7.

Theorem 11.21 (Lovász). $\Theta(C_5) = \sqrt{5}$.

Most of the material in this course regarding the Shannon capacity is from the first two pages of that paper. Today we shall also state a result from page 4.

But first, back to page 2: we prove Theorem 11.21. All we need is find an ONR of C_5 and a handle such that the value is $\sqrt{5}$. This implies that $\vartheta(C_5) \le \sqrt{5}$ and therefore

$\Theta(C_5) \leq \sqrt{5}$. Given that we already know the inequality $\Theta(C_5) \geq \sqrt{5}$ (recall that this follows from the exercise that we can place 5 kings on the 5×5 toroidal chessboard), we conclude that $\Theta(C_5) = \sqrt{5}$ and incidentally the ONR and handle we found are optimal. But for we do not need to prove optimality in advance, we just need to guess the right ONR and handle. This choice will be quite natural from symmetry considerations.

So here is the construction of the **Lovász umbrella** for C_5 in $\mathbb{R}^3$. First draw a regular pentagon in the unit circle in the XY plane, centered around the origin $\varnothing = (0,0,0)$ and with one of its vertices the point $\mathbf{u}_1 = (1,0,0)$, the others numbered clockwise as $\mathbf{u}_2, \ldots, \mathbf{u}_5$. Pick a point $\mathbf{p} = (0,0,z)$ on the Z axis, $z \geq 0$. Note that $\mathbf{p} \perp \mathbf{u}_i$.

We choose z such that $\mathbf{p} - \mathbf{u}_1 \perp \mathbf{p} - \mathbf{u}_3$. Let $\mathbf{a}_i$ be the unit vector in the direction $\mathbf{p} - \mathbf{u}_i$:

$$\mathbf{a}_i = \frac{\mathbf{p} - \mathbf{u}_i}{\|\mathbf{p} - \mathbf{u}_i\|}. \tag{4}$$

We take $\{\mathbf{a}_1, \ldots, \mathbf{a}_5\}$ for our ONR and we choose the handle to be the unit vector $\mathbf{c}$ in the direction $-\mathbf{p}$:

$$\mathbf{c} = \frac{-\mathbf{p}}{\|p\|}. \tag{5}$$

This is illustrated in Figure 2.

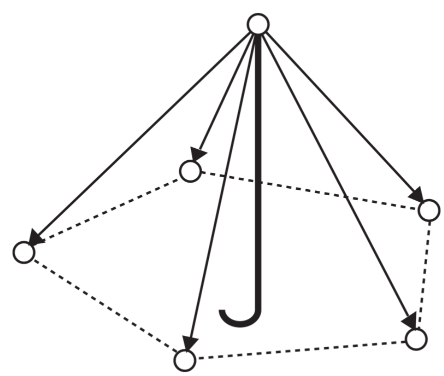

Figure 2: The Lovász umbrella for C_5. Image: cs.chalmers.se

The condition $\mathbf{p} - \mathbf{u}_1 \perp \mathbf{p} - \mathbf{u}_3$ together with the 5-fold rotational symmetry of the configuration about the handle ensures that this is indeed an ONR of C_5. Next we determine the right value of z to achieve this.

DO 11.22. $z^2 = \cos(\pi/5)$.

Proof. Recall that $\mathbf{p} = (0,0,z)$ be the tip of the umbrella and $\mathbf{u}_1 = (1,0,0)$. Observe that $\mathbf{u}_3 = (\cos(4\pi/5), \sin(4\pi/5), 0)$. We wish to have

$$\begin{aligned} 0 &= (\mathbf{p} - \mathbf{u}_1)^T(\mathbf{p} - \mathbf{u}_3) \\ &= \mathbf{p}\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{u}_1 - \mathbf{p}\cdot\mathbf{u}_3 + \mathbf{u}_1\cdot\mathbf{u}_3 \\ &= z^2 - 0 - 0 + \mathbf{u}_1\cdot\mathbf{u}_3 \\ &= z^2 + \cos(4\pi/5). \end{aligned}$$

Finally, note that $\cos(4\pi/5) = -\cos(\pi/5)$. □

Let us now evaluate our ONR and handle.

Evaluating the Lovász umbrella. We have

$$\frac{1}{(\mathbf{c}^T\mathbf{a}_1)^2} = \frac{\|\mathbf{p}\|^2 \|\mathbf{p}-\mathbf{u}_1\|^2}{(\mathbf{p}^T(\mathbf{p}-\mathbf{u}_1))^2} = \frac{z^2(z^2+1)}{z^4}$$
$$= 1 + \frac{1}{z^2} = 1 + \frac{1}{\cos(\pi/5)} = \sqrt{5}.$$

In the last line we used exercise DO 11.3. ☐

HW 11.23. (6 points) Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis of $\mathbb{R}^3$. Consider the following list of vectors: $\mathbf{a}_1 = \mathbf{e}_1$, $\mathbf{a}_2 = \mathbf{e}_1$, $\mathbf{a}_3 = \mathbf{e}_2$, $\mathbf{a}_4 = \mathbf{e}_2$, $\mathbf{a}_5 = \mathbf{e}_3$. Most of you verified that $(\mathbf{a}_1, \dots, \mathbf{a}_5)$ form an ONR of C_5. Find the optimal handle to this ONR (the handle that minimizes the value) and compute the value of this ONR with its optimal handle. Prove that your handle is best possible.

GOOD CHARACTERIZATION OF ϑ

Lovász also obtained the following remarkable dual characterization of the ϑ function. This result, combined with the definition of ϑ, provide a "good characterization" for $\vartheta(G)$.

Theorem 11.24.

$$\vartheta(G) = \max_{(\mathbf{w}_i),\mathbf{d}} \sum_{i=1}^{n} (\mathbf{d}^T\mathbf{w}_i)^2$$

where the maximum is taken over all choices $(\mathbf{w}_i)_{i=1}^n$ of an ONR of $\overline{G}$ and handle $\mathbf{d}$.

Note that this statement links $\vartheta(G)$ to the ONRs of the **complement** of G.

We shall not prove this result, but we illustrate its power on some consequences.

DO 11.25. $\vartheta(G * H) \geq \vartheta(G) \cdot \vartheta(H)$.

Corollary 11.26. $\vartheta(G * H) = \vartheta(G) \cdot \vartheta(H)$.

Another consequence will be that ϑ beats L-dim as an upper bound for the Shannon capacity.

Theorem 11.27. $\vartheta(G) \leq$ L-dim(G).

Lemma 11.28. *If $\mathbf{a}_1, ..., \mathbf{a}_n$ are an ONR of G and $\mathbf{c}$ is a handle, and $\mathbf{b}_1, ..., \mathbf{b}_n$ are an ONR of $\overline{G}$ with $\mathbf{d}$ a handle, then*

$$\sum_{i=1}^{n} (\mathbf{c}^T\mathbf{a}_i)^2(\mathbf{d}^T\mathbf{b}_i)^2 \leq 1.$$

Proof. Based on Lemma 11.10 and the observation that for any i, j, either $\mathbf{a}_i \perp \mathbf{a}_j$ or $\mathbf{b}_i \perp \mathbf{b}_j$, it follows that $\{\mathbf{a}_i \otimes \mathbf{b}_i \mid i = 1, ..., n\}$ is an ON system. Thus

$$1 = \|\mathbf{c}\|^2 \|\mathbf{d}\|^2 = \|(\mathbf{c} \otimes \mathbf{d})\|^2 \geq \sum_{i=1}^{n} ((\mathbf{c}\otimes\mathbf{d})^T(\mathbf{a}_i \otimes \mathbf{b}_i))^2$$
$$= \sum_{i=1}^{n} (\mathbf{c}^T\mathbf{a}_i)^2(\mathbf{d}^T\mathbf{b}_i)^2.$$

The inequality in the first line was based on Parseval's inequality. ☐

Combined with Theorem 11.20, this result gives the following corollary.

Corollary 11.29. *If* $\mathbf{b}_1, \dots, \mathbf{b}_n$ *is an ONR of* $\overline{G}$ *with* $\mathbf{d}$ *a handle, then* $\vartheta(G) \geq \sum_{i=1}^{n} (\mathbf{d}^T \mathbf{b}_i)^2$.

This proves the "easy" direction (min $\geq$ max) of the good characterization of ϑ.

Proof. Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be an ONR and of G and $\mathbf{c}$ a handle with equal angles, i. e., $\mathbf{c}^T \mathbf{a}_i = 1/\sqrt{\vartheta}$ for all i. Now apply Lemma 11.28. □

HW 11.30. (6 points) Prove that $\vartheta(G) \cdot \vartheta(\overline{G}) \geq n$. Use any of the stated results without proof; state what you use. Your proof should not be more than a few lines.

HW 11.31. (4 points) Prove: for perfect graphs, $\alpha(G) \cdot \alpha(\overline{G}) \geq n$. (2 lines.)

HW 11.32. (2 points) Find a graph for which $\alpha(G) \cdot \alpha(\overline{G}) < n$.

CH 11.33. (20 points) Find a graph for which $\alpha(G) \cdot \alpha(\overline{G}) < n/100$.

HW 11.34 (Due Tuesday; please do not hand in before Tuesday)**. (5 points)**
Prove: $\alpha^*(G) \cdot \alpha^*(\overline{G}) \geq n$.

HW 11.35 (Due Tuesday; please do not hand in before Tuesday)**. (6 points)**
Prove: $\vartheta(G) \leq \text{L-dim}(G)$.
Hint. Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be an ONR of G in $\mathbb{R}^d$ where $d = \text{L-dim}(G)$. Prove that the vectors $\mathbf{a}_1 \otimes \mathbf{a}_1, \dots, \mathbf{a}_n \otimes \mathbf{a}_n$ also form an ONR of G (in dimension d^2); denote it by ONR_2. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be an ONB in $\mathbb{R}^d$. Let

$$\mathbf{g} = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} \mathbf{e}_j \otimes \mathbf{e}_j.$$

Prove that this is a unit vector. Take $\mathbf{g}$ as the handle with ONR_2. Prove that the value of this pair is $\leq d$.