Graph Theory: CMSC 27530/37530 Lecture 12

Lecture by László Babai
Notes by Geoffrey West
Revised by instructor

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EXTREMAL GRAPH THEORY

Recall our terminology: the *order* of a graph is the number of its vertices; the *size* of a graph is the number of edges. Extremal graph theory studies the maximum size of graphs of a given order, having a given property. The most extensively studied properties are *forbidden subgraphs.*

Definition 12.1. For a graph H we write $\mathrm{ex}(n, H)$ to denote the maximum size (number of edges) among the graphs of order n containing no subgraph isomorphic to H.

For instance, the Mantel–Turán theorem states that

$$\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor. \tag{1}$$

Turán's theorem gives the exact value of $\mathrm{ex}(n, K_{t+1})$ for all $t \geq 2$ with the consequence that

$$\mathrm{ex}(n, K_{t+1}) \leq \left(1 - \frac{1}{t}\right) \frac{n^2}{2}. \tag{2}$$

Turán's theorem also describes the *extremal graphs* – the graphs that have size $\mathrm{ex}(n, K_{t+1})$.

DO 12.2. For all $t \geq 2$ show that inequality (2) is asymptotically tight, i. e.,

$$\mathrm{ex}(n, K_{t+1}) \sim \left(1 - \frac{1}{t}\right) \frac{n^2}{2}. \tag{3}$$

It is easy to show that $\mathrm{ex}(n, C_5) \geq \mathrm{ex}(n, K_3)$; in fact, the two quantities are asymptotically equal as $n \to \infty$, but this is much harder to prove.

HW 12.3. (7 points) Let H be a graph and let $s = \chi(H) \geq 3$. Prove: $\mathrm{ex}(n, H) \geq \mathrm{ex}(n, K_s)$. You may use Turán's Theorem. State exactly what you use.

Next we state the fundamental theorem of extremal graph theory which states that the inequality $\mathrm{ex}(n, H) \geq \mathrm{ex}(n, K_{\chi(H)})$ is asymptotically tight for all graph H with chromatic number ≥ 3.

CH* 12.4 (Erdős–Stone–Simonovits)**.** If $\chi(H) \geq 3$, then $\mathrm{ex}(n, H) \sim \mathrm{ex}(n, K_{\chi(H)})$.

The essence of the result was proved by Pál [Paul] Erdős and Arthur H. Stone in 1946 and two decades later put in this striking form by Erdős and Miklós Simonovits.

Why do we exclude the case when $\chi(H) = 2$? Note that $\mathrm{ex}(n, K_2) = 0$. The case of bipartite graphs H was first studied by Tamás Kővári, Vera T. Sós, and Pál [Paul] Turán in a 1954 paper. Here is the simplest case of their result.

Theorem 12.5 (Kővári–Sós–Turán)**.**

$$\mathrm{ex}(n, C_4) < \frac{n^{3/2}}{2} + \frac{n}{4}. \tag{4}$$

In particular, $\mathrm{ex}(n, C_4) = O(n^{3/2})$.

Proof. Let N denote the number of copies of P_3 (paths of length 2) contained in the graph G as subgraphs. We count the copies of P_3 in two ways: by their center and by their endpoints. Let $V = [n]$ and for $i \in V$ let $d_i = \deg(i)$. Vertex i is the center of $\binom{d_i}{2}$ copies of P_3, so $N = \sum_{i=1}^{n} \binom{d_i}{2}$. On the other hand, consider a pair $\{u, v\}$ of distinct vertices. There cannot be more than one path of length 2 connecting them since two such paths would create a C_4 subgraph. It follows that $N \le \binom{n}{2}$ and therefore

$$\binom{n}{2} \ge N = \sum_{i=1}^{n} \binom{d_i}{2}.$$

Multiplying each side by 2,

$$\begin{aligned} n(n-1) &\ge \sum_{i=1}^{n} d_i(d_i - 1) = \sum_{i=1}^{n} (d_i^2 - d_i) \\ &\ge \frac{(\sum_{i=1}^{n} d_i)^2}{n} - \sum_{i=1}^{n} d_i = \frac{(2m)^2}{n} - 2m. \end{aligned}$$

The second inequality in this series uses the inequality between the arithmetic and quadratic means. The last equality uses the Handshake theorem. Now we have

$$\begin{aligned} n^3 &= n^2(n-1) + n^2 \\ &\ge 4m^2 - 2mn + n^2 > \left(2m - \frac{n}{2}\right)^2. \end{aligned}$$

The result follows by taking the square root. □

DO 12.6. $\mathrm{ex}(n, K_{2,100}) = O(n^{3/2})$.

BONUS 12.7. (6 points) $\mathrm{ex}(n, K_{3,3}) = O(n^{5/3})$.

CH* 12.8. Show that the asymptotic bound in the preceding problem is tight. In other words, find a constant $c > 0$ and inifinitely many graphs without $K_{3,3}$ subgraphs such that $m \ge cn^{5/3}$ for these graphs.

The general bound for forbidden complete bipartite subgraphs given by Kővári, Sós, and Turán is the following.

DO 12.9. For $2 \le s \le t$ we have

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-\frac{1}{s}}). \tag{5}$$

This bound is tight for $s2, 3$. It is a long-standing open problem to decide whether this bound is tight for any value $s \ge 3$. The best lower bound known for the case of $K_{s,s}$ for $s \ge 4$ is

$$\mathrm{ex}(n, K_{s,s}) = \Omega(n^{2-\frac{2}{s}}). \tag{6}$$

LINEAR ALGEBRA

Remark 12.10. A **closed-form expression** is a formula that is built from variables and constants using basic functions referred to as "primitives" without the use of $\sum$, $\prod$, or "..." notation. The primitives include the four arithmetic operations (addition, subtraction, multiplication, division), taking powers, logarithms, and rounding (the floor and ceiling functions). In particular, the trigonometric functions and the Fibonacci numbers are included (why?). Additionally, in this class, we include factorials and therefore binomial coefficients among our primitives. So for instance

$$\lfloor \sqrt{F_{n-2}} \rfloor + \log_n \binom{n^2}{n-1} \cdot \tan(\pi/(n!))$$

is a closed-form expression.

HW 12.11. (6 points) Consider the $n \times n$ matrix

$$A = \begin{pmatrix} a & & & & \\ & a & & \text{\Large b} & \\ & & a & & \\ & \text{\Large b} & & \ddots & \\ & & & & a \end{pmatrix}$$

So, letting $A = (a_{ij})$, we have $a_{ii} = a$ and $a_{ij} = b$ for $i \neq j$.
Compute the determinant $\det(A)$ as a simple closed-form expression in the variables a, b, and n.

Notation 12.12. $\mathbb{R}^{k \times n}$ denotes the set of $k \times n$ matrices over $\mathbb{R}$, i. e., $k \times n$ matrices with real entries. Due to the special role of square matrices, $\mathbb{R}^{n \times n}$ is also denoted $M_n(\mathbb{R})$. For matrices with complex entries, the analogous notation $\mathbb{C}^{k \times n}$ and $M_n(\mathbb{C}) = \mathbb{C}^{n \times n}$ is used.

Definitions 12.13. For a matrix $A \in M_n(\mathbb{R})$, an **eigenvector** of A is a **non-zero** vector $\mathbf{x} \in \mathbb{R}^n$ such that $(\exists \lambda \in \mathbb{R})(A\mathbf{x} = \lambda \mathbf{x})$.
The number λ is the **eigenvalue** corresponding to the eigenvector $\mathbf{x}$. The number $\lambda \in \mathbb{R}$ is an eigenvalue of the matrix A if there exists a corresponding eigenvector.

The lower case Greek letter λ is called "lambda" (LaTeX: (`\lambda`)).

Definition 12.14. A polynomial $f(t) = \sum_{i=0}^{n} a_i t^i = a_n t^n + \ldots + a_1 t + a_0$ in the variable t has **degree** n if $a_n \neq 0$. In this case we refer to $a_n t^n$ as the *leading term* and a_n as the *leading coefficient*. f is a **monic polynomial** if its leading coefficient is $a_n = 1$.

Recall the Kronecker delta symbol δ_{ij}. It has value 1 if $i = j$ and zero if $i \neq j$. The **identity matrix** is the $n \times n$ matrix $I = (\delta_{ij})$.

Definition 12.15. The **characteristic polynomial** of the $n \times n$ matrix A is

$$f_A(t) = \det(tI - A)$$

where I is the $n \times n$ identity matrix.

DO 12.16. The characteristic polynomial is a monic polynomial of degree n.

DO 12.17. Let I be the $n \times n$ identity matrix. Then

$$f_I(t) = (t-1)^n. \tag{7}$$

Definition 12.18. For an $n \times n$ matrix A, the **trace** of A is the sum of the diagonal entries:

$$\mathrm{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$

Example 12.19. Let A be the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then A has the characteristic polynomial

$$\begin{aligned} f_A(t) = \begin{vmatrix} t-a & -b \\ -c & t-d \end{vmatrix} &= (t-a)(t-d) - bc = t^2 - (a+d)t + (ad - bc) \\ &= t^2 - \mathrm{trace}(A) \cdot t + \det(A). \end{aligned}$$

Remark 12.20. In the definitions of eigenvalues, eigenvectors, characteristic polynomial, we can replace $\mathbb{R}$ by $\mathbb{C}$ and get the analogous concepts over the complex numbers. A *polynomial over* $\mathbb{R}$ is a polynomial with real coefficients; a *polynomial over* $\mathbb{C}$ is a polynomial with complex coefficients. Remember that $\mathbb{R} \subset \mathbb{C}$, i. e., every real number is a complex number (with imaginary part equal to zero). So, in particular, $M_n(\mathbb{R}) \subset M_n(\mathbb{C})$.
In the context of vector spaces, the numbers permitted as entries in a matrix or as coefficients in a linear combination (real or complex) are called **scalars**. So the set of scalars is $\mathbb{R}$ or $\mathbb{C}$.

DO 12.21. The eigenvalues are precisely the roots of the characteristic polynomial. In other words, λ is an eigenvalue of A if and only if $f_A(\lambda) = 0$.

Corollary 12.22. *An $n \times n$ matrix has at most n eigenvalues.*

Theorem 12.23 (Fundamental Theorem of Algebra)**.** *If f is a monic polynomial of degree n over $\mathbb{C}$ then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that*

$$f(t) = \prod_{i=1}^{n} (t - \lambda_i).$$

Definition 12.24. The λ_i are the **roots** of f (also called the **zeros** of f). The **multiplicity** of the scalar $\alpha \in \mathbb{C}$ among the roots of f is the number of subscripts i such that $\alpha = \lambda_1$. A **simple root** is a root with multiplicity is 1; a **multiple root** is a root with multiplicity is ≥ 2.

Example 12.25. If $f(t) = (t-1)^5(t+2)^3(t+8)$, then the roots of f are $1, -2$ and -8 with respective multiplicities $5, 3$, and 1. So 1 and -2 are multiple roots, while-8 is a simple root.

DO 12.26. Let f be a polynomial over $\mathbb{C}$. The value $\alpha \in \mathbb{C}$ is a multiple root of f if and only if $f(\alpha) = f'(\alpha) = 0$. Here f' is the derivative of f. Solve this over $\mathbb{R}$ if you are not comfortable with complex numbers.

Definition 12.27. The **multiplicity of eigenvalue** λ for the $n \times n$ matrix A is the multiplicity of λ in the characteristic polynomial f_A.

DO 12.28. The $n \times n$ identity matrix has the number 1 as its sole eigenvalue; it has multiplicity n.

DO 12.29. Let $f_A(t) = \det(tI - A) = \prod_{i=1}^n (t - \lambda_i)$.

(a) $\text{trace}(A) = \sum_{i=1}^n \lambda_i$.

(b) $\det(A) = \prod_{i=1}^n \lambda_i$.

HW 12.30. (4 points) Let $A \in \mathbb{R}^{k\times n}$ and $B \in \mathbb{R}^{n\times k}$. Prove: $\text{trace}(AB) = \text{trace}(BA)$.

Definition 12.31. A list of vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$, is **linearly independent** if

$$(\forall \alpha_1, \ldots, \alpha_n \in \mathbb{R}) \left(\sum_{i=1}^n \alpha_i \mathbf{v}_i = 0 \iff (\forall i)(\alpha_i = 0) \right).$$

In other words, a list of vectors is linearly independent if the trivial linear combination is the only linear combination of the list that evaluates to the zero vector. (A linear combination $\sum_i \alpha_i \mathbf{v}_i$ is the **trivial linear combination** if all coefficients α_i are zero.)

Definition 12.32. The **dimension** of a vector space V is the maximum number of linearly independent vectors in V. This number is denoted $\dim(V)$.

Remark 12.33. If you are not comfortable with the abstract definition of a vector space, think of V as either $\mathbb{R}^n$ or a subspace of $\mathbb{R}^n$, here as well as later in these notes.)

DO 12.34. $\dim(\mathbb{R}^n) = n$. This central fact of linear algebra is not straightforward; you need to prove that any list of $n+1$ vectors in $\mathbb{R}^n$ is linearly dependent. If you get stuck, look up a proof.

HW 12.35. (4 points) If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is an orthonormal system, then it is linearly independent.

HW 12.36. (6 points) Let $A \in M_m(\mathbb{R})$. If $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are eigenvectors corresponding to distinct eigenvalues, then they are linearly independent. (The condition says that the $\mathbf{a}_i$ are nonzero vectors such that $A\mathbf{a}_i = \lambda_i$ where for $i \neq j$ we have $\lambda_i \neq \lambda_j$.)

The converse does not hold: linearly independent eigenvectors may or may not share the same eigenvalue.

DO 12.37. Show that all non-zero vectors are eigenvectors of the identity matrix.

Definition 12.38. A **scalar matrix** is a matrix of the form λI where λ is a number.

DO 12.39. If A is a scalar matrix then all nonzero vectors are eigenvectors of A.

BONUS 12.40. (6 points) Let $A \in M_n(\mathbb{R})$. Prove: if every nonzero vector $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of $A \in M_n(\mathbb{R})$ then A is a scalar matrix.

Definition 12.41. We say that a list L of vectors **spans** the vector space V is every vector in V is a linear combination of L.

Definition 12.42. A list L of vectors is a **basis** of the vector space V if

1. L is linearly independent
2. L spans V.

DO 12.43. A list L of vectors is a basis for a vector space V if and only if every vector in V is **uniquely expressible** as a linear combination of L. In other words, the list $L = (\mathbf{b}_i)_{i=1}^n$ is a basis if and only if

$$(\forall \mathbf{w} \in V)(\exists! \text{ scalars } \alpha_1, \ldots, \alpha_n) \left(\mathbf{w} = \sum_{i=1}^n \alpha_i \mathbf{b}_i \right).$$

Notation 12.44. The phrase "with respect to" is a preposition which requires three separate words–what an extroardinary excess! We abbreviate the phrase by writing "wrt".

Definition 12.45. For a vector $\mathbf{w}$ and a basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$, the **coordinates** of $\mathbf{w}$ wrt the basis $(\mathbf{b}_i)_{i=1}^n$ are the unique scalars $(\alpha_1, \ldots, \alpha_n)$ such that $\mathbf{w} = \sum_{i=1}^n \alpha_i \mathbf{b}_i$.

Definition 12.46. An **eigenbasis** of $A \in M_n(\mathbb{R})$ is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Remark 12.47. WARNING. An eigebasis of $A \in M_n(\mathbb{R})$ is NOT a "basis of A" but a basis of the space $\mathbb{R}^n$. In particular, it consists of n vectors, regardless of the rank of A.

DO 12.48. Find an eigenbasis of the $n \times n$ all-zero matrix. Find all eigenbases.

Definition 12.49. A matrix $A \in M_n(\mathbb{R})$ is **diagonalizable** over $\mathbb{R}$ if A has an eigenbasis.

Remark 12.50. As usual, we have the analogous definitions over $\mathbb{C}$.
In the next exercises we shall encounter real matrices that are diagonalizable over $\mathbb{C}$ but not diagonalizable over $\mathbb{R}$.

Definition 12.51. For $\theta \in \mathbb{R}$, the **rotation matrix** A_θ is defined as

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Remark 12.52 (Why "rotation matrix?"). One can use coordinates wrt a given basis to establish a one-to-one correspondence between linear transformations of $\mathbb{R}^n$ and the matrices in $M_n(\mathbb{R})$. The matrix A_θ turns out to be the matrix corresponding to the rotation of the plane by the angle θ or $-\theta$ wrt any ONB of the plane. (When is it θ and when is it $-\theta$? Note that the ONBs of the plane can naturally be divided into two kinds. (Explain.)

HW 12.53. ♡ **(5 points)** Find the characteristic polynomial of A_θ. Find its complex roots.

Remark 12.54. The ♡ indicates that this is a sweet problem. In this case, we start by turning the geometry (rotation of the plane) into algebra, then a polynomial, then its roots), and the output (a pair of complex numbers) turns out to be intimately related to the geometry we started with (rotation). Question: how are the complex numbers you get related to the rotation of the plane by angle θ ?
You find many sweet problems on the Puzzle Problems sheet.

BONUS 12.55. (4 points) Find a **complex** eigenbasis of the rotation matrix A_θ. So, even though A_θ is a real matrix, you need to view it as a complex matrix.

DO 12.56. If $\theta \neq k\pi$, then A_θ has no real eigenvalues, and therefore no real eigenvectors.

HW 12.57. (Due Thursday) **(6 points)** Let $A \in M_n(\mathbb{R})$ be the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Find the characteristic polynomial of A.

BONUS 12.58. (Due Thursday) **(2+5 points)** A the same as in the preceding problem, viewed as a complex matrix.

(a) Find the eigenvalues of A over $\mathbb{C}$.

(b) Find an eigenbasis of A over $\mathbb{C}$.

HW 12.59. (5 points) The matrix $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no eigenbasis over $\mathbb{R}$, and no eigenbasis over $\mathbb{C}$. (The first statement follows from the second, but ignore the second if you don't like complex numbers.)

Definition 12.60. $A = (a_{ij})$ is a **triangular matrix** if $i > j \implies a_{ij} = 0$. In picture,

$$A = \begin{pmatrix} * & & & & \\ & * & & \huge{*} & \\ & & * & & \\ & \huge{0} & & \ddots & \\ & & & & * \end{pmatrix}.$$

(The stars indicate arbitrary entries, possibly including zero)

DO 12.61. If A is a triangular matrix, then

$$f_A(t) = \prod_{i=1}^{n}(t - a_{ii}).$$

In particular, the eigenvalues of a diagonal matrix are its diagonal elements.

Definition 12.62. Let f be a monic polynomial of degree n with real coefficients. We say that **all roots of** f **are real** if $f(t) = \prod_{i=1}(t - \lambda_i)$ where all the λ_i are real.
We say that **all eigenvalues of** $A \in M_n(\mathbb{R})$ **are real** if all roots of the characteristic polynomial f_A are real.

DO 12.63. All eigenvalues of a real triangular matrix are real.

Definition 12.64 (Interlacing)**.** Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1}$, where the λ_i and the μ_j are real numbers. We say the two sequences **interlace** if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \lambda_n.$$

We also say that the μ_j interlace the λ_i.

HW 12.65. (Due Thursday) **(6 points)** Let f be a monic polynomial of degree n over $\mathbb{R}$. Suppose all roots of f are real. Prove:

(a) All roots of f' are real.

(b) The roots of f and f' interlace.

SPECTRAL THEOREM, QUADRATIC FORMS, INTERLACING EIGENVALUES We now state one of the central results of linear algebra.

Theorem 12.66 (The Spectral Theorem)**.** *If $A \in M_n(\mathbb{R})$ is a symmetric real matrix (i. e., $A = A^T$), then A has an orthonormal eigenbasis over $\mathbb{R}$. In particular, all eigenvalues of A are real.*

While in the results stated so far, $\mathbb{R}$ can simply be replaced by $\mathbb{C}$ to obtain an analogous result for complex numbers, the situation is not so simple in this case. There is an analogous result for complex Hermitian matrices which we shall state later.

The Spectral Theorem is said to be the second-most often applied theorem of (higher) mathematics, in terms of applications both within mathematics and to the sciences. (It is preceded by the Fundamental Theorem of Calculus.) In particular, graph theory is a voracious consumer of the Spectral Theorem.

DO 12.67. If $A \in \mathbb{R}^{k\times n}$ and $\mathbf{x} \in \mathbb{R}^k$, $\mathbf{y} \in \mathbb{R}^n$, then

$$\mathbf{x}^T A\mathbf{y} = \sum_{i=1}^{k}\sum_{j=1}^{n} a_{ij}x_iy_j.$$

If $A \in M_n(\mathbb{R})$ is symmetric, then it follows from the previous exercise that

$$\mathbf{x}^T A\mathbf{x} = \sum_{i,j} a_{ij}x_i x_j = \sum_i a_{ii}x_i^2 + 2\sum_{i<j} a_{ij}x_i x_j. \tag{8}$$

An expression of this form is called a **quadratic form**, also known as a "homogenous polynomial of degree 2."

Definition 12.68. An expression of the form $\sum_{i=1}^n \alpha_i x_i$ is a **linear form** (where the x_i are variables and the α_i are scalars). These are exactly the homogeneous linear polynomials.

In the problems below, use the Spectral Theorem.

HW 12.69. (Due Thursday) **(5 points)** Let $A \in M_n(\mathbb{R})$ be a real symmetric matrix. Let $\mathbf{b}_1, \dots, \mathbf{b}_n$ be an orthonormal eigenbasis of A with corresponding eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. (So $A\mathbf{b}_i = \lambda_i \mathbf{b}_i$.) Let $\mathbf{x} \in \mathbb{R}^n$ be expressed as $\mathbf{x} = \sum_{i=1}^n \beta_i \mathbf{b}_i$ (so $\beta_1, \dots, \beta_n$ are the coordinates of $\mathbf{x}$ wrt the basis $(\mathbf{b}_i)_{i=1}^n$). Then

$$\mathbf{x}^T A\mathbf{x} = \sum_{i=1}^n \lambda_i \beta_i^2. \tag{9}$$

Definition 12.70. If $A \in M_n(\mathbb{R})$ is a symmetric real matrix, then the **Rayleigh quotient** of A is the function

$$R_A(\mathbf{x}) = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$$

defined for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$.

DO 12.71. $R_A(\alpha\mathbf{x}) = R_A(\mathbf{x})$ whenver $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

HW 12.72 (Rayleigh's Principle)**.** (Due Thursday) **(5 points)**
Let $A \in M_n(\mathbb{R})$ be a real symmetric matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$.

(i) $\lambda_1 = \max_{\mathbf{x}} R_A(\mathbf{x})$.

(ii) $\lambda_n = \min_{\mathbf{x}} R_A(\mathbf{x})$.

Lord *Rayleigh* (1842–1919) was a British physicist. He spent his entire academic career at Cambridge. He won the Nobel prize in physics in 1904 and was cited, among other things, for the discovery of the noble gas argon. His actual last name is Strutt; his full name is *John William Strutt*, 3rd Baron Rayleigh.

BONUS 12.73 (Courant–Fischer)**.** (Due Thursday) **(8 points)** Let $A \in M_n(\mathbb{R})$ be a symmetric real matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Then, for all k $(1 \leq k \leq n)$,

$$\lambda_k = \max_{\substack{U \leq \mathbb{R}^n \\ \dim U = k}} \min_{\substack{\mathbf{x} \in U \\ \mathbf{x} \neq 0}} R_A(\mathbf{x})$$

Here the maximum ranges over all k-dimensional subspaces of $\mathbb{R}^n$.

Richard Courant (1888–1972) was a German–Jewish–American mathematician. Hover and click here to read his fascinating biography, his rise from a kid who at some point in elementary school was rated "less than satifactory" in arithmetic, to a prominent German mathematician, even while his father's business went into bankruptcy at Richard's age of 14 from which time on he had to support himself fully by tutoring. At 19 he was admitted to the University of Göttingen, Germany, one of the world centers of mathematics at the time, and within a year became David Hilbert's assistant. Hilbert was considered *the* leading mathematician of the time. Courant received his PhD under Hilbert at the age of 22. Much of Courant's work was in mathematical physics. He was wounded in WWI. From 1918 he was a member of the faculty at Göttingen and founded the Mathematical Institute there. In 1933 the Nazis came to power, Courant was driven from his position and had to leave the country. In 1934 he moved to New York. At NYU, Courant almost single-handedly created a first-class mathematical institute, now named the Courant Institute.

Ernst S. Fischer (1875–1954) was a mathematician born in Vienna. Austria. He spent most of his career in Erlangen, Germany. His work was in analysis, specifically in orthogonal functions. His most famous result is the Riesz–Fischer theorem, asserting the completeness of the L^2 space, a fundamental result in the theory of Hilbert spaces and at the foundations of the field of functional analysis. The result was proved in 1907 independently by Fischer and Hungarian mathematician Frigyes (Frederic) Riesz (1880–1956).

BONUS 12.74 (Interlacing eigenvalues)**.** (Due Thursday) **(8 points)** Let $A \in M_n(\mathbb{R})$ be a symmetric real matrix. Let $B \in M_{n-1}(\mathbb{R})$ be the matrix obtained by deleting the i-th row and the i-th column of A, as shown below.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}}_{A} \to \underbrace{\begin{pmatrix} a_{11} & \cancel{a_{12}} & \dots & a_{1n} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{\dots} & \cancel{a_{2n}} \\ \vdots & \cancel{\vdots} & \ddots & \vdots \\ a_{n1} & \cancel{a_{n2}} & \dots & a_{nn} \end{pmatrix}}_{B}$$

So B is also a symmetric real matrix. Prove that the eigenvalues of B interlace the eigenvalues of A. (Hint: Courant–Fischer.)