Graph Theory: CMSC 27530/37530 Lecture 14

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Revised by instructor

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Please remember to **send the instructor the list of challenge problems you solved** (except 6.23: $\alpha(C_7 * C_7)$) so he can check if his records are complete.

FRACTIONAL INDEPENDENCE NUMBER

A HW problem asked to prove $\alpha^*(G) \cdot \alpha^*(\overline{G}) \geq n$.

Proof. We give a simple proof of the stronger statement

$$\alpha^*(G) \cdot \alpha(\overline{G}) \geq n. \tag{1}$$

For the graph $G = (V, E)$, recall the LP that defines the fractional independence number $\alpha^*(G)$. We associate the variable x_v with vertex v and impose the following constraints:

(1) $(\forall v \in V)(x_v \geq 0)$

(2) $(\forall \text{ clique } C \text{ in } G)(\sum_{v \in C} x_v \leq 1)$.

We seek to maximize $\sum_{v \in V} x_v$ under these constraints.

In order to give a lower bound on this maximum, $\alpha^*(G)$, it suffices to guess a feasible solution. Let us set $x_v = \dfrac{1}{\alpha(\overline{G})} = \dfrac{1}{\omega(G)}$ for each $v \in V$. This is a feasible solution, i. e., it satisfies the constraints. (Verify this!) Therefore

$$\alpha^*(G) \geq \sum_{v \in V} x_v = \frac{n}{\alpha(\overline{G})} \ . \tag{2}$$

Note that we did not use the LP Duality Theorem for this proof. □

This was a one-line proof; all that matters is contained in this line.

For $v \in V$ let $x_v = \dfrac{1}{\alpha(\overline{G})}$. This is a feasible solution. Therefore, $\alpha^*(G) \geq \displaystyle\sum_{v \in V} x_v = \frac{n}{\alpha(\overline{G})}$.

CHROMATIC POLYNOMIAL

BONUS 14.1 (Due Thursday)**. (6 points)** The chromatic polynomial has no roots in the open interval $(0, 1)$.

RAMSEY THEORY

Let's play the **Ramsey Game** on six vertices: We have a Red player and a Blue player. The players alternate selecting edges of K_6. Every edge is selected only once, so the game is over in $\binom{6}{2} = 15$ rounds. A player loses if their selected edges contain a triangle.

Theorem 14.2. *No draw is possible in the Ramsey Game on six vertices.*

Proof. We prove the following stronger statement.

Claim 14.3 (Baby Ramsey Theorem). No matter how we color $E(K_6)$ red and blue, there exists a monochromatic triangle. ("Monochromatic" means all edges have the same color.)

Choose a vertex u. It has degree 5 in K_6. At least three of the edges from u must have the same color; let's say $\{u, v_i\}$ are red for $i = 1, 2, 3$. One of two things must be true.

1. $\{v_i, v_j\}$ is red for some $\{i, j\} \subset \{1, 2, 3\}$
2. all the three edges $\{v_i, v_j\}$ are blue ($\{i, j\} \subset \{1, 2, 3\}$).

In the first case, $\{u, v_i, v_j\}$ is a red triangle. In the second case, $\{v_1, v_2, v_3\}$ is a blue triangle. □

Notation 14.4 (Erdős–Rado arrow symbol)**.** We write $n \to (k, \ell)$ if

$$(\forall \text{ Red/Blue coloring of } E(K_n))(\exists \text{ Red } K_k \text{ or } \exists \text{ Blue } K_\ell).$$

Examples. $n \to (n, 2)$, $6 \to (3, 3)$, $5 \nrightarrow (3, 3)$ (prove!)

BONUS 14.5 (Erdős–Szekeres, 1934)**. (6 points)** For $k, \ell \geq 1$ we have $\binom{k+\ell}{k} \to (k+1, \ell+1)$.

Use induction on $k + \ell$. The base cases are $k = 1$ or $\ell = 1$ (infinitely many base cases!); for the inductive step you may then assume $k, \ell \geq 2$.

Setting $k = 2$, $\ell = 2$ we obtain $6 \to (3, 3)$ (the baby case). This is tight: $5 \nrightarrow (3, 3)$.
Setting $k = 3$, $\ell = 2$ we obtain $10 \to (4, 3)$. This can be improved.

BONUS 14.6 (Due Thursday)**. (6 points)** $9 \to (4, 3)$.

HW 14.7. (5 points) $17 \to (3, 3, 3)$. Define the arrow symbol for this case. (Use three colors.)

DO 14.8. $4^k > \binom{2k}{k}$. Use $2^n = \sum_{i=0}^{n} \binom{n}{i}$.

But 4^k is not much bigger than $\binom{2k}{k}$.

HW 14.9. (4 points)

$$\frac{\binom{2k}{k}}{4^k} \sim \frac{c}{\sqrt{k}}.$$

Determine the constant c. Use **Stirling's formula**, the most famous asymptotic equality:

$$n! \sim \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n} \ . \qquad (3)$$

Notation 14.10 (Diagonal case of the arrow symbol)**.** We write $n \to (k)_2$ for $n \to (k, k)$ and $n \to (k)_3$ for $n \to (k, k, k)$, etc.

From the Erdős–Szekeres Theorem we get

$$\binom{2k}{k} \to (k+1)_2 \tag{4}$$

Combining this with the inequality $4^k > \binom{2k}{k}$ we obtain

$$4^k \to (k+1)_2 \tag{5}$$

or, writing $n = 2^k$,

$$n \to \left(1 + \frac{1}{2}\log_2 n\right)_2 \tag{6}$$

QUESTION 14.11. *How far is this from best possible? In other words, can we estimate the smallest value of k such that $n \nrightarrow (k)_2$?*

To better understand this question, let us rephrase the meaning of the arrow notation. Given a graph $G = (V, E)$, let us say that a subset $A \subseteq V$ is **homogeneous** if A is either a clique or an independent set in G.

DO 14.12. The statement $n \to (k, \ell)$ is equivalent to the following:

For all graphs G with n vertices we have

$$\omega(G) \geq k \quad \text{or} \quad \alpha(G) \geq \ell \ . \tag{7}$$

In particular, the statement $n \to (k)_2$ is equivalent to saying that

every graph on n vertices has a homogeneous subset of size k.

Erdős showed (1949) that for all sufficiently large n,

$$n \nrightarrow (1 + 2\log_2 n)_2. \tag{8}$$

Comparing this with Eq. (6) we see a gap of 4 between the upper and lower bounds. These bounds have been known for 70 years, yet nobody has been able to reduce the gap of 4 by any constant amount (say to 3.99). This remains one of the great **open questions** in graph theory and in Ramsey theory.

Integrality gap. Erdős's result (Eq. (8)) tells us that there exist graphs that simultaneously satisfy

$$\alpha(G) = O(\log n) \quad \text{and} \quad \alpha(\overline{G}) = O(\log n) \ . \tag{9}$$

In particular, such graphs satisfy

$$\alpha(G) \cdot \alpha(\overline{G}) = O((\log n)^2) \ . \tag{10}$$

Contrast this with the result we proved at the beginning of this class:

$$\alpha^*(G) \cdot \alpha(\overline{G}) \ge n \ . \tag{11}$$

So for Erdős's graphs we have $\alpha(G) = O(\log n)$ while $\alpha^*(G) = \Omega(n/\log n)$, a huge "integrality gap." Moreover, Erdős's bounds hold for **almost all graphs** (they hold for random graphs with probability approaching 1 as $n \to \infty$), which shows that α^* is an extremely poor approximation to α for *most graphs.*

DO 14.13. Prove: for all sufficiently large n we have $\quad (\log_2 n)^{100} < n$.

Proof of existence vs. explicit construction. Erdős's result says that there exist graphs without a homogenous subset of size $1 + 2\log_2 n$. But Erdős did not construct such graphs. In an early display of the power of his **probabilistic method**, he just proved that such graphs exist, by proving that almost all graphs have this property. The next question is, construct **explicit graphs** with only very small homogeneous subsets.

HW 14.14. (4 points) Give a constructive proof of the relation $k^2 \not\to (k+1)_2$. In other words, for all k, construct a graph with k^2 vertices that does not have a homogeneous subset of size $k+1$.

This will show that $n \not\to (1+\sqrt{n})_2$ for infinitely many values of n (namely, the values $n = k^2$).

CH 14.15 (H. L. Abbott)**. (8 points)** Give a constructive proof of the relation
$5^k \not\to (2^k+1)_2$. Hint: invent another graph product. Don't look it up.

This will show that $n \not\to (1 + n^{\log 2/\log 5})_2$ for infinitely many values of n (verify!).
Since $\log 2/\log 5 \approx 0.43$, this is an improvement over exercise 14.14.

POLYNOMIALS OF MATRICES

DO 14.16. Let $A \in M_n(\mathbb{C})$. If $\lambda \in \operatorname{spec}(A)$ then $\lambda^2 \in \operatorname{spec}(A^2)$.

Proof. Let $\mathbf{x}$ be an eigenvector to eigenvalue λ, so $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \lambda\mathbf{x}$. Then
$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. □

HW 14.17. (5 points) Let $A \in M_n(\mathbb{C})$. If g is a polynomial and $\lambda \in \operatorname{spec}(A)$, then $g(\lambda) \in \operatorname{spec}(g(A))$.

HW 14.18. (5 points) Let g be a polynomial. If A is a diagonalizable matrix and $\operatorname{spec}(A) = \{\{\lambda_1, \ldots, \lambda_n\}\}$, then $g(A)$ is also diagonalizable and $\operatorname{spec}(g(A)) = \{\{g(\lambda_1), \ldots, g(\lambda_n)\}\}$.

CH 14.19. (6 points) Over $\mathbb{C}$ every matrix is similar to a triangular matrix. Do not use Jordan normal form.

BONUS 14.20. (6 points) Use the preceding problem to show that over $\mathbb{C}$ the same relation as in problem 14.18 holds between the spectrum of A and the spectrum of $g(A)$ regardless of the diagonalizability of A. In other words, prove the following. If $A \in M_n(\mathbb{C})$ and $\operatorname{spec}(A) = \{\{\lambda_1, \ldots, \lambda_n\}\}$, then $\operatorname{spec}(g(A)) = \{\{g(\lambda_1), \ldots, g(\lambda_n)\}\}$.

CH 14.21. (4 points) Diagonalizable matrices are dense in $M_n(\mathbb{C})$. Use any reasonable metric.

GRAPH SPECTRA

Recall that for a graph $G = ([n], E)$, the **adjacency matrix** $A_G = (a_{ij})$ is the $n \times n$ matrix defined by

$$a_{ij} = \begin{cases} 1 & i \sim j \\ 0 & \text{o/w} \end{cases}$$

In particular, $a_{ii} = 0$. An important observation about the adjacency matrix is that it is symmetric: $A_G = A_G^T$. This permits us to apply the Spectral Theorem to it; this will be our basic tool.

In particular, the eigenvalues of A_G are real; we shall list them in decreasing order:

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \ . \tag{12}$$

Notation 14.22. For a graph G, we speak of the **spectrum of the graph**, meaning the spectrum of its adjacency matrix: $\operatorname{spec}(G) := \operatorname{spec}(A_G)$.

DO 14.23.

$$\sum_{i=1}^{n} \lambda_i(G) = 0 \ . \tag{13}$$

DO! 14.24. Let $A_G^k = (a_{ij}^{(k)})$. Then $a_{ij}^{(k)} = \#$ of $i ... j$ walks of length k.

Let us look at the trace of the powers of A_G. We have $\operatorname{trace}(A_G) = 0$ because $a_{ii} = 0$.

DO 14.25. $\operatorname{trace}(A_G^2) = 2m$. Hint. $a_{ii}^{(2)} = \deg(i)$.

HW 14.26. (5 points) What is $\operatorname{trace}(A_G^3)$? Explain the answer in terms of counting certain subgraphs.

A previous challenge problem stated the following. If t_G is the number of triangles in G and m_G is the number of edges, then

$$t_G \leq \frac{\sqrt{2}}{3} m_G^{3/2}. \tag{14}$$

We have also seen that for $G = K_n$ we have LHS $\sim$ RHS (previous HW).

BONUS 14.27 (Due Thursday)**. (7 points)** Prove inequality (14). Use only the tools from class.

Once done with this problem, take a moment to marvel at the power of linear algebra. Naturally, this problem ceases to be a challenge problem.

The effect of transforming a vector x by the adjacency matrix.
Let $\mathbf{y} = A_G \mathbf{x}$. Then each entry y_i in the vector $\mathbf{y}$ has a simple form:

$$y_i = \sum_{j \,:\, j \sim i} x_j \ . \tag{15}$$

DO 14.28. Verify Eq. (15).

What is the effect on the all-ones vector?

DO 14.29.

$$A_G \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \deg(1) \\ \deg(2) \\ \vdots \\ \deg(n) \end{pmatrix}.$$

DO 14.30. If G is r-regular, meaning $(\forall v)(\deg(v) = r)$, then r is an eigenvalue of G.

In fact, it is the largest eigenvalue. This follows from the following exercise.

HW 14.31. (6 points) For every graph G, $(\forall i)(\,|\lambda_i(G)| \le \deg_{\max})$.

HW 14.32. (6 points) Prove:

$$\lambda_1(G) \ge \frac{\sum_{i=1}^n \deg(i)}{n}. \tag{16}$$

Hint. Give a one-line solution using Rayleigh's Principle.

CH 14.33. (9 points) Prove:

$$\lambda_1(G) \ge \sqrt{\frac{\sum_{i=1}^n \deg(i)^2}{n}}. \tag{17}$$

In the light of the inequality between the arithmetic mean and quadratic mean, this lower bound is stronger than Eq. (16).

BONUS 14.34. (5 points) Use Eq. (17) to prove that equality holds in Eq. (16) if and only if G is regular.

HW 14.35 (Herbert Wilf, 1961)**. (7 points)** Prove: $\chi(G) \le 1 + \lambda_1(G)$.

In the light of exercise 14.31, this result strengthens the easy upper bound $\chi(G) \le 1+\deg_{\max}$.

Notation 14.36. The characteristic polynomial of a graph G is $f_G := f_{A_G}$.

DO 14.37. If a matrix A has the 2×2 block-triangular form

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

where the diagonal blocks A_{11} and A_{22} are square matrices then $\det(A) = \det(A_{11}) \cdot \det(A_{22})$ and consequently $f_A = f_{A_{11}} \cdot f_{A_{22}}$.

This works for $k \times k$ block-triangular matrices as well. Here is a picture of the 3×3 case.

$$A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline 0 & A_{22} & A_{23} \\ \hline 0 & 0 & A_{33} \end{array}\right]$$

In this case, $\det(A) = \det(A_{11}) \cdot \det(A_{22}) \cdot \det(A_{33})$ and consequently $f_A = f_{A_{11}} \cdot f_{A_{22}} \cdot f_{A_{33}}$.

We use this to reduce the determination of the charactristic polynomial of a matrix to its connected components.

Denote a disconnected graph G by $G = H_1 \sqcup H_2 \sqcup \ldots \sqcup H_k$ where the H_i are the connected components. Then A_G has the block-diagonal form $\text{diag}(A_{H_1}, \ldots, A_{H_k})$, illustrated here in the $k = 3$ case.

$$A_G = \left[\begin{array}{c|c|c} A_{H_1} & 0 & 0 \\ \hline 0 & A_{H_2} & 0 \\ \hline 0 & 0 & A_{H_3} \end{array}\right].$$

It follows by the lemma that $f_G(t) = f_H(t) \cdot f_L(t)$.

DO 14.38. If $G = H_1 \sqcup H_2 \sqcup \ \ldots \ \sqcup H_k$ where the H_i are the connected components of G, then $\lambda_1(G) = \max(\lambda_1(H_i) \mid i = 1, \ldots, k)$.

DO 14.39. If G is r-regular and has k connected components then $\lambda_1 = \cdots = \lambda_k = r$.

We shall show that $\lambda_1 = \lambda_2$ can only occur for disconnected graphs.

Theorem 14.40. *If G is connected then $\lambda_2 < \lambda_1$.*

This condition is not "if and only if."

HW 14.41. (5 points) Find a disconnected graph G with $\lambda_1 = 87$ and $\lambda_2 = 14$.

RAYLEIGH'S PRINCIPLE REVISITED

Recall that for $A \in M_n(\mathbb{R})$, the function $R_A : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$, called the *Rayleigh quotient* of A, is defined by

$$R_A(\mathbf{x}) = \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

DO 14.42. Prove that the $\mathbb{R}_A$ function has a maximum value. Do not use the Spectral Theorem.

DO 14.43. Prove: if $\mathbf{v}$ is an eigenvector of A to eigenvalue μ then $R_A(\mathbf{v}) = \mu$.

What we previously stated as "Rayleigh's Principle" is only part of the story. Here is a more complete form.

Theorem 14.44 (Rayleigh's Principle)**.** *Let $A \in M_n(\mathbb{R})$. Let $\lambda = \max\limits_{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}} R_A(\mathbf{x})$. If $\mathbf{u} \in \mathbb{R}^n$ satisfies $R_A(\mathbf{u}) = \lambda$ then $\mathbf{u}$ is an eigenvector.*

DO 14.45. Show that the eigenvalue corresponding to the vector $\mathbf{u}$ in Theorem 14.44 is necessarily λ, and λ is the largest real eigenvalue of A. Do not use the Spectral Theorem.
Hint. Use Exercise 14.43.

DO 14.46. Use Theorem 14.44 to prove the same result regarding the minimum value of R_A and the smallest real eigenvalue of A. Do not use the Spectral Theorem.
Hint. Apply Theorem 14.44 to the matrix $-A$.

Remark 14.47. The significance of not using the Spectral Theorem in several of the problems above is that a simple inductive proof of the Spectral Theorem is based on an elegant direct proof of Rayleigh's Principle.

CH 14.48. (6 points) Give a direct proof of Rayleigh's Principle. Do not use the Spectral Theorem. Do not hand in your solution if you looked it up.
Hint. Let $\mathbf{u}$ be a vector that maximizes the Rayley quotient. (Why does such a vector exist?) Show that $\mathbf{u}$ is an eigenvector. To prove this, let $\mathbf{v} \perp \mathbf{u}$. Consider the function $h(t) = R_A(\mathbf{u} + t\mathbf{v})$ ($t \in \mathbb{R}$). Use the fact that this function attains its maximum at $t = 0$.

MORE SPECTRAL GRAPH THEORY

We define the Rayleigh quotient of a graph G as $R_G = R_{A_G}$.

Theorem 14.49. *$\lambda_1(G)$ has a non-negative eigenvector.*

Proof. Let $\mathbf{x} = (x_1, \dots, x_n)^T$ be an eigenvector to eigenvalue λ_1; therefore $R_G(\mathbf{x}) = \lambda_1$ by Exercise 14.43. Let $\widetilde{\mathbf{x}} = (|x_1|, \dots, |x_n|)$. Then

$$\lambda_1 \geq R_G(\widetilde{\mathbf{x}}) \geq R_G(\mathbf{x}) \geq \lambda_1 \ . \tag{18}$$

(Why?) So we have $R_G(\widetilde{\mathbf{x}}) = \lambda_1$. Therefore, by Rayleigh's Principle, $\widetilde{\mathbf{x}}$ is an eigenvector to eigenvalue λ_1 (see Remark 14.47). □

BONUS 14.50. (7 points) Assume G is connected. Let $\mathbf{x} = (x_1, \dots, x_n)$ be an eigenvector to λ_1. Then either all the x_i are positive or all the x_i are negative.

DO 14.51. If G is connected, then λ_1 is unique.

Proof. Suppose $\mathbf{u}$ and $\mathbf{v}$ are two linearly independent eigenvectors to eigenvalue λ_1. Then every nontrivial linear combination of $\mathbf{u}$ and $\mathbf{v}$ is also an eigenvector to λ_1 (why?). Among these one can find a vector $\mathbf{w}$ that is orthogonal to $\mathbf{u}$. Now either all coordinates of $\mathbf{u}$ are positive or all are negative by Problem 14.50, and the same holds for $\mathbf{w}$. But two such vectors cannot be orthogonal. (Why?) □

Corollary 14.52. *If G is connected then $\lambda_2(G) < \lambda_1(G)$.*

Indeed, this is just a restatement of the uniqueness of λ_1.

HW 14.53. (6 points) If G is bipartite, then $\operatorname{spec}(G) = -\operatorname{spec}(G)$.
What this means is that $\lambda_n = -\lambda_1$, $\lambda_{n-1} = -\lambda_2, \dots,$ i. e., $\lambda_{n-i} = -\lambda_{i+1}$ for every i.

BONUS 14.54 (Due Thursday)**. (7 points)** If G is connected and $\lambda_n = -\lambda_1$, then G is bipartite.

CH 14.55. (8+8 points) (a) Prove: If G is connected and has diameter d then G has at least $d + 1$ distinct eigenvalues. (b) This bound is tight for the d-cube Q_d.